# Lattice QCD with Chemical Potential

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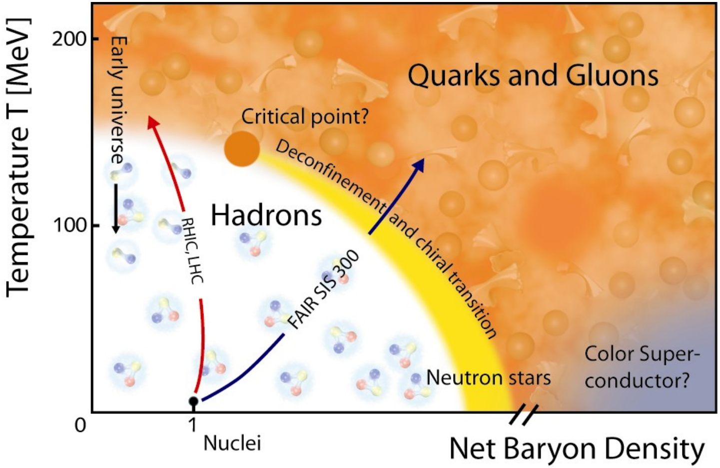
# INTEREST IN HIGH TEMPERATURES AND HIGH MATTER DENSITIES

- Better understanding of quark-gluon matter shortly after Big Bang
- Neutron stars: High densities and temperatures
- Heavy ion collisions: Matter under extreme conditions

# THE QCD PHASE DIAGRAM



# THE QCD PHASE DIAGRAM

- Transition from a confined to a deconfined phase
- For zero baryon density at about 170 $MeV$ $\longrightarrow$ crossover
- Increasing baryon density $\rightsquigarrow$ crossover turns into a first order phase transition
- For $m_u = m_d$: Critical endpoint of the first order transition line is of second order
- For $m_u = m_d = m_s$: The first order line extends until the point $\mu = 0$

# QCD ON A SPACE-TIME LATTICE

- Put continuous space-time on a lattice
- The lattice is defined as:

$$\Lambda = \{\mathbf{x} = a \cdot \mathbf{n} | \mathbf{n} = (n_1, n_2, n_3, n_4)\}$$

- Continuum fermion action:

$$S_F[\psi, \overline{\psi}, A] = \int d^4x \, \overline{\psi}(x) D \psi(x)$$

$$D = \gamma_\mu [\partial_\mu + i A_\mu(x)] + m$$

- Discretisation prescription:

$$\begin{aligned} \psi(\mathbf{x}), \overline{\psi}(\mathbf{x}) &\longrightarrow \psi(\mathbf{n}), \overline{\psi}(\mathbf{n}) \\ \int d^4x \, \ldots &\longrightarrow a^4 \sum_{\mathbf{n} \in \Lambda} \ldots \\ \partial_\mu \, \psi(\mathbf{x}) &\longrightarrow \frac{\psi(\mathbf{n} + \hat{\mu}) - \psi(\mathbf{n} - \hat{\mu})}{2a} + \mathcal{O}(a^2) \end{aligned}$$

# THE LATTICE ACTION

$$S[U,\overline{\psi},\psi] = S_F[U,\overline{\psi},\psi] + S_G[U]$$

- Fermion action $S_F$:

$$S_F[U,\overline{\psi},\psi] = \sum_{f=1}^{N_f} a^4 \sum_{n,m\in\Lambda} \overline{\psi}^{(f)}(n) D^{(f)}(n,m) \psi^{(f)}(m)$$

- Partition function:

$$Z = \int \mathcal{D}[\overline{\psi},\psi]\mathcal{D}[U]\ e^{-S_G[U]-S_F[\overline{\psi},\psi]}$$

- Integrate out fermion fields:

$$Z = \int \mathcal{D}[\overline{\psi},\psi]\mathcal{D}[U]\ e^{-S_G[U]-\overline{\psi}D\psi}$$

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$$Z = \int \mathcal{D}[U]\ e^{-S_G[U]} \det[D]$$

# LATTICE SIMULATION

- Evaluation of $Z$ with Monte Carlo techniques is possible
- $\det D \in \mathbb{R}$
- $\Longrightarrow$ real Boltzmann weight
- Generate configurations according to the weight factor $P[U]$:

$$P[U] = e^{-S_G[U]} \det[U]$$

# Introduction of chemical potential

- Grand-canonical ensemble:

$$Z_{GC}(\mu) = \mathrm{Tr}[e^{-(H-\mu N_q)/T}]$$

- Path integral formulation:

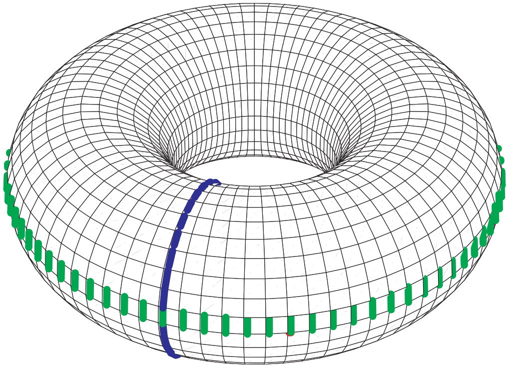
$$Z_{GC}(\mu) = \int \mathcal{D}[U,\overline{\psi},\psi]\; e^{-S[U,\overline{\psi},\psi]}$$

- The Dirac matrix with chemical potential:

$$\begin{aligned} D(n,m)_{\alpha\beta,ab} &= \left(\frac{4}{a}+m\right)\delta_{\alpha\beta}\delta_{ab}\delta_{nm} - \\ & \frac{1}{2a}\sum_{\nu=\pm 1}^{\pm 4} f(\mu)\,(\mathbb{1}-\gamma_\nu)_{\alpha\beta}\; U_\nu(n)_{ab}\;\delta_{n+\hat{\nu},m} \end{aligned}$$

$$f(\mu) = \begin{cases} 1 & : \quad \hat{\nu} = 1,2,3,-1,-2,-3 \\ e^{a\mu} & : \quad \hat{\nu} = 4 \\ e^{-a\mu} & : \quad \hat{\nu} = -4 \end{cases}$$

# INTRODUCTION OF CHEMICAL POTENTIAL



- At each temporal hop the Dirac matrix picks up a factor $e^{\mu}$
- $N_T$ temporal lattice points
- Move chemical potential at the temporal boundary of the lattice $\Longrightarrow e^{\mu N_T}$

# $\gamma_5$-HERMITICITY

- Introduction of $\mu$ causes a severe technical problem
- $\Longrightarrow$ Dirac operator is no longer $\gamma_5$-hermitian
- $\gamma_5$-hermiticity:

$$\gamma_5 D \gamma_5 = D^\dagger \longrightarrow \det[D] = \det[D]^*$$

- Including the chemical potential:

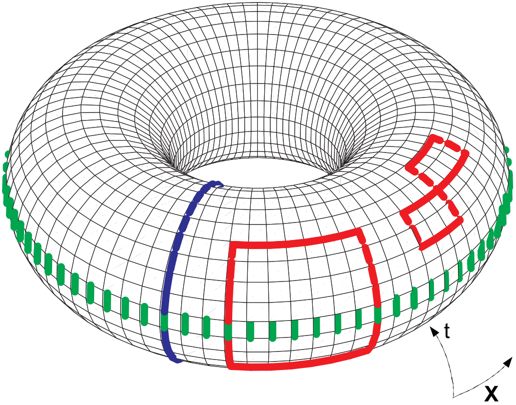
$$\gamma_5 D(\mu) \gamma_5 = D^\dagger(-\mu)$$

- Hence the fermion determinant is complex:

$$\det[D(\mu)] \neq \det[D(\mu)]^*$$

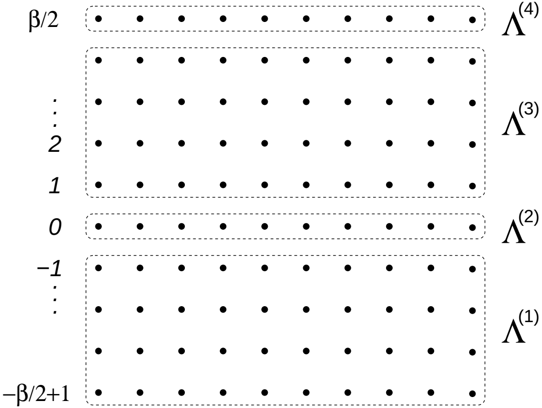
$\longrightarrow$ fermion sign problem
$\longrightarrow$ Standard Monte Carlo techniques fail

# FERMION DETERMINANT




- Write fermion determinant as a sum of loops
- Only the loops that wind around time cause a problem!
- There are forward and backward winding loops $\Longrightarrow$ complex number

# Lattice decomposition



# FACTORISATION OF THE DETERMINANT

- Rewrite the fermion determinant:

$$\det[D] = \int \mathcal{D}[\overline{\psi}, \psi]\; e^{-\overline{\psi} D \psi}$$

- Use the partition of the lattice into domains
- Integrate out the Grassmann variables $\overline{\psi}^{(i)}, \psi^{(i)}$ in each domain $\Lambda^{(i)}$
- Result:

$$\det[D] = A_0 W$$

$$A_0 = \det\left[D^{(1)}\right] \det\left[D^{(3)}\right] \det\left[\widetilde{D}^{(2)}\right] \det\left[\widetilde{D}^{(4)}\right]$$

$$W = \det\left[1 - \widetilde{S}^{(4)}\left[e^{\mu\beta}\widetilde{D}_1^{(4,2)} + \widetilde{D}_3^{(4,2)}\right]\widetilde{S}^{(2)}\left[e^{-\mu\beta}\widetilde{D}_1^{(2,4)} + \widetilde{D}_3^{(2,4)}\right]\right]$$

# DECOMPOSITION INTO WINDING SECTORS

- Expand $W$ according to its number of loops
- Trace-log formula:

$$\det[\mathbb{1} - M] = \exp(\mathrm{Tr}[\ln(\mathbb{1} - M)]) = \exp\left( -\sum_{n=1}^{\infty} \frac{1}{n} \mathrm{Tr}[M^n] \right)$$

- Introduce:

$$\begin{aligned} H_0 &= \widetilde{S}^{(4)} D_1^{(4,2)} \widetilde{S}^{(2)} D_1^{(2,4)} + \widetilde{S}^{(4)} D_3^{(4,2)} \widetilde{S}^{(2)} D_3^{(2,4)} \\ H_{+1} &= \widetilde{S}^{(4)} D_1^{(4,2)} \widetilde{S}^{(2)} D_3^{(2,4)} \\ H_{-1} &= \widetilde{S}^{(4)} D_3^{(4,2)} \widetilde{S}^{(2)} D_1^{(2,4)} \end{aligned}$$

- Dimensionally reduction:

$$W = \det\left[\mathbb{1} - H_0 - e^{\mu\beta} H_{+1} - e^{-\mu\beta} H_{-1}\right]$$

# STATISTICAL PHYSICS

Partition function $Z$ encodes the statistical properties of a system.

- **Canonical partition function $Z_C^{(Q)}$:**

  Thermodynamic system in a heat bath with temperature $T$, volume and particle number $Q$ fixed

- **Grand-canonical partition function $Z_{GC}(\mu)$:**

  System can exchange heat and particles with the environment, which has temperature $T$ and chemical potential $\mu$

- **Fugacity expansion:**

$$Z_{GC}(\mu) = \sum_Q e^{\beta\mu Q} Z_C^{(Q)}$$

# $D^{(Q)}$ from the Fourier transformation method

- Fourier transformation:

$$Z_C^{(Q)} = \frac{1}{2\pi} \int d\varphi e^{-iQ\varphi}\, Z_{GC}(\mu = i\varphi/T)$$

- Insert $Z_{GC}$:

$$Z_C^{(Q)} \;=\; \int \mathcal{D}[U]\, e^{-S_G[U]}\, D^{(Q)}$$

- Projected determinants $D^{(Q)}$:

$$D^{(Q)} \;=\; \int_{-\pi}^{\pi} \frac{d\varphi}{2\pi}\, e^{-iQ\varphi}\, \det[D(\mu = i\varphi/T)]$$

- Use factorisation formula for the determinant:

$$D^{(Q)} = \frac{A_0}{2\pi} \int_{-\pi}^{\pi} d\varphi\, e^{-iQ\varphi} \det[\mathbb{1} - H_0 - e^{i\varphi} H_{+1} - e^{-i\varphi} H_{-1}]$$

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# FREE ENERGY

- Free Energy of a single quark:

$$\frac{Z^{(Q=1)}}{Z^{(Q=0)}} = \frac{\langle \det[Q=1] \rangle}{\langle \det[Q=0] \rangle} = e^{-\beta F}$$

- Confinement/Deconfinement:

$$\frac{Z^{(Q=1)}}{Z^{(Q=0)}} = \begin{cases} 0 & : \quad T < T_c \implies F = \infty \\ \neq 0 & : \quad T > T_c \implies F < \infty \end{cases}$$

[Gattringer, Liptak: arXiv:0906.1088, 2009]

# Center symmetry

- $SU(N)$ gauge theory has symmetry under center transformation:

$$U_4(\vec{x}, t_0) \longrightarrow z\, U_4(\vec{x}, t_0)$$

- $SU(3)$: $Z_3$ center

$$z \in \{1, e^{2i\pi/3}, e^{-2i\pi/3}\}$$

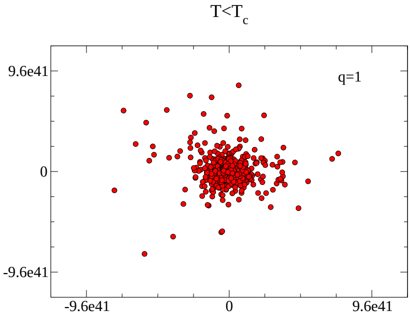
- Center transformation:

$$\det[D]^{(Q)} \longrightarrow z^Q \det[D]^{(Q)}$$

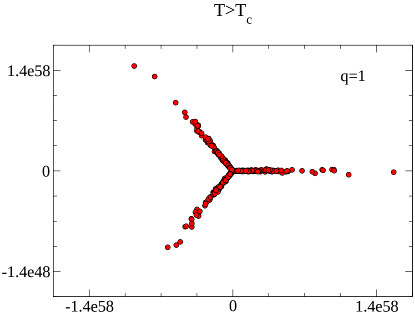
- Consequence for canonical partition functions:

$$Z^{(Q)} = \langle \det[D]^{(Q)} \rangle_G = z^Q \langle \det[D]^{(Q)} \rangle_G$$

$\Longrightarrow Z^{(Q)}$ vanishes in the unbroken phase, unless $z^Q = 1$



$T<T_c$
q=1
9.6e41
0
-9.6e41
-9.6e41
0
9.6e41

$T>T_c$
q=1
1.4e58
0
-1.4e48
-1.4e58
0
1.4e58

# SUMMARY

- We investigated the fermion action with chemical potential
- Found the fermion determinant in a sector with fixed quark number $\longrightarrow$ Canonical approach
- Idea:
  - Partition of the lattice
  - Fermion determinant expressed in terms of subdeterminants
  - Only last subdeterminant part contains the chemical potential!!
  - Apply a winding number decomposition

# SUMMARY

- Idea allows to calculate the canonical determinant at a fixed quark number
- Factorisation of the determinant dimensionally reduces the part which depends on $\mu$
- Study of free energy $\Longrightarrow$ phase transitions
- Application: Simulations with fixed quark number are possible