Renormalisation Group Flows of QCD in Coulomb Gauge in a Hamiltonian Formulation

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Outline

Introduction

- Development of QED: Perturbative renormalisation
- Wilson: renormalisation group in statistical physics
- Wetterich: Introduction of the average effective action: non-perturbative renormalisation: FRG

  C. Wetterich, Phys.Lett.B301:90-94,1993

  Prominent application: possible non-perturbative renormalisability of GR: Quantum Einstein Gravity (Reuter)

The Generating Functional Z_k

$$Z[j] = \int \mathcal{D}\chi \, e^{-S[\chi] + j\cdot\chi}$$

The Generating Functional Z_k

$$Z_k[j] = \int \mathcal{D}_\Lambda \chi \, e^{-S[\chi] - \Delta S_k[\chi] + j \cdot \chi}$$

The Generating Functional Z_k

$$Z_k[j] = \int \mathcal{D}_\Lambda \chi \, e^{-S[\chi]-\Delta S_k[\chi]+j\cdot\chi}$$

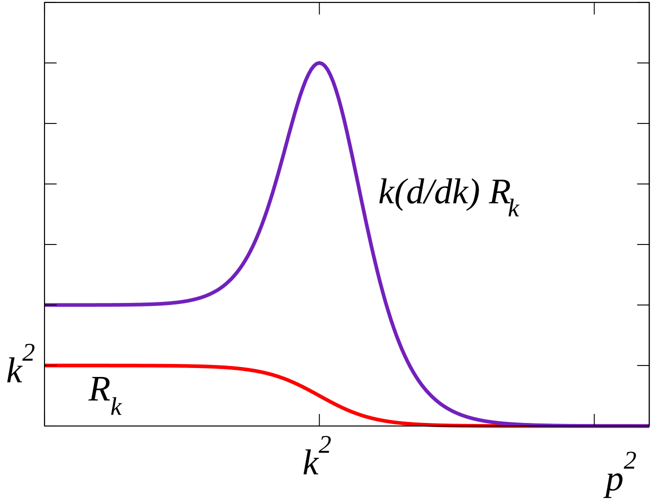
$$\Delta S_k[\chi] = \frac{1}{2}\chi \cdot R_k \cdot \chi$$

IR-regularisation (mass term): $\lim_{p^2/k^2 \to 0} R_k(p) > 0$

recovering full theory: $\lim_{k^2/p^2 \to 0} R_k(p) = 0$

input: bare action $(\Gamma_{\Lambda\to\infty} \to S)$ $\lim_{k^2 \to \Lambda \to \infty} R_k(p) \to \infty$

A Typical Regulator $R_k(p)$



$Z_k \longrightarrow W_k \longrightarrow \Gamma_k$: The FRG for Γ_k

The Functional Renormalisation Group Equation

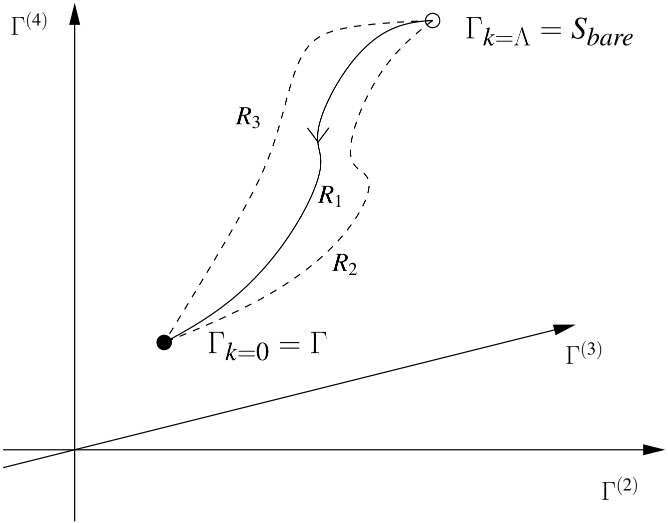
$$\partial_t \Gamma_k[\phi] = \frac{1}{2} \mathrm{Tr} \left[\partial_t R_k \left(\Gamma_k^{(2)}[\phi] + R_k \right)^{-1} \right]$$

C. Wetterich, Phys. Lett. **B301** 90 (1993)

$$\text{where } \Gamma_k^{(2)}[\phi] = \frac{\delta^2 \Gamma_k[\phi]}{\delta\phi\,\delta\phi} \quad \text{and} \quad k\frac{d}{dk} =: \partial_t$$

- Γ_k interpolates between $\Gamma_{k=\Lambda} = S_{bare}$ and $\Gamma_{k=0} = \Gamma$.
- Physics for $k > \Lambda$ is regarded as included already in Γ_Λ.
- Therefore, cutoff Λ is NOT taken to ∞, Λ stays large but finite.

Theory Space



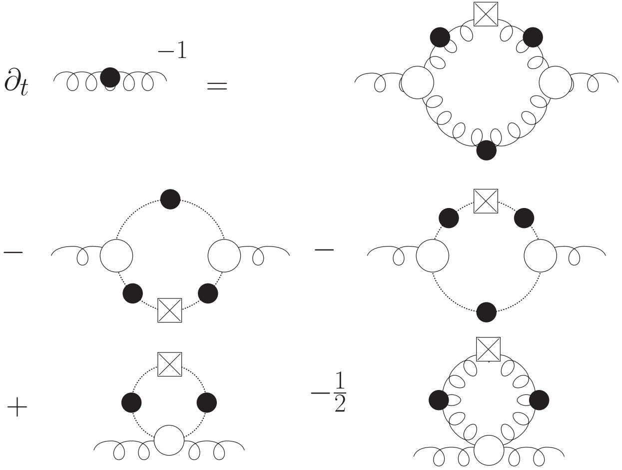
Definition of Z_k

- Define the theory in a gauge fixed way:
  Weyl gauge ($A_0^a = 0$) and Coulomb gauge ($\partial_i A_i^a = 0$).
- VEV is taken wrt. the full ground state.

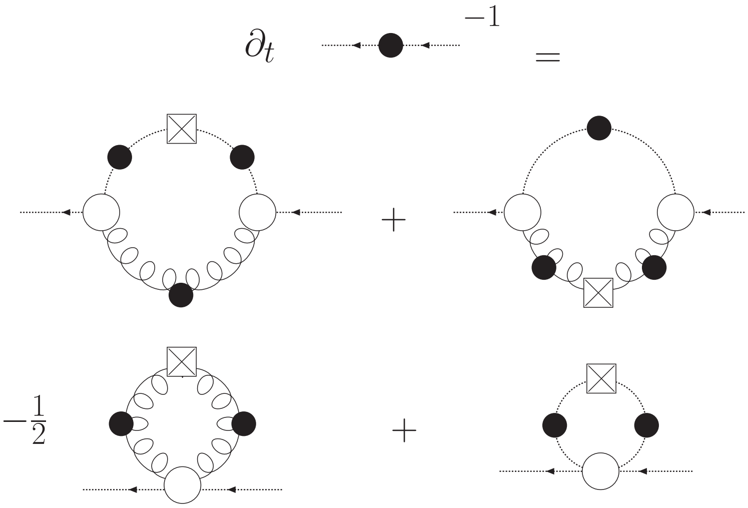
$$
\begin{aligned}
Z_k[J, \sigma, \bar{\sigma}] &= \langle \psi | \exp[-\Delta S_k + J \cdot A + \bar{\sigma} \cdot c + \bar{c} \cdot \sigma] | \psi \rangle \\
&= \int \mathcal{D}[A, c, \bar{c}] \exp[-S - \Delta S_k + J \cdot A + \bar{\sigma} \cdot c + \bar{c} \cdot \sigma] \\
\Delta S_k &= \tfrac{1}{2} A \cdot R_k \cdot A + \bar{c} \cdot \tilde{R}_k \cdot c
\end{aligned}
$$

- No ansatz for the vacuum wave functional is used.
- No Hamiltonian is used.

The Flow of the Gluon Propagator



The Flow of the Ghost Propagator



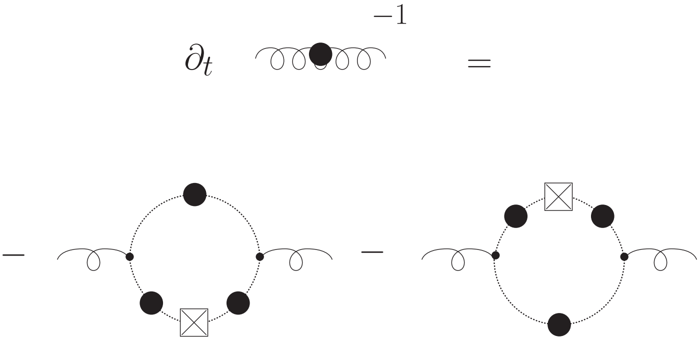
Truncation of the Theory Space

Restriction of theory space to

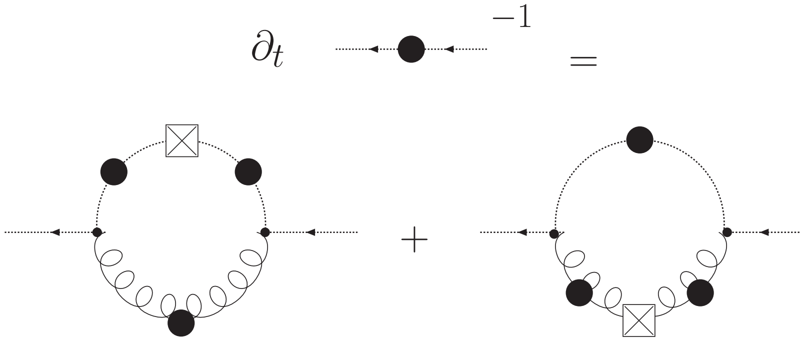
- gluon propagator
- ghost propagator
- BARE ghost-gluon vertex

$\Rightarrow$ flow equations decoupled from the rest of theory space

Truncated Gluon Flow



Truncated Ghost Flow



Decomposition

Weyl gauge $\Rightarrow$ only spatial fields $A^a_{x,y,z}$, $A^a_0 = 0$
Coulomb gauge $\Rightarrow$ transversality $t_{ij}(\mathbf{p})$ of gluon propagator

$$\frac{\delta^2 \Gamma_k[0]}{\delta A^a_i(\mathbf{p})\ \delta A^b_j(\mathbf{q})} = \delta^{ab} t_{ij}(\mathbf{p}) 2\omega_k(p)(2\pi)^d \delta^d(\mathbf{p}+\mathbf{q})$$

$$\frac{\delta^2 \Gamma_k[0]}{\delta c^a(\mathbf{p})\ \delta \bar{c}^b(\mathbf{q})} = \delta^{ab} \frac{p^2}{d_k(p)} (2\pi)^d \delta^d(\mathbf{p}+\mathbf{q})$$

$$\frac{\delta^3 \Gamma_\Lambda}{\delta A^a_i(\mathbf{p})\ \delta c^b(\mathbf{q})\ \delta \bar{c}^c(\mathbf{k})} = f^{cab}(iq_j) t_{ij}(\mathbf{p})(2\pi)^d \delta^d(\mathbf{p}+\mathbf{q}+\mathbf{k})$$

Renormalisation Conditions

How to implement renormalisation conditions?

- Ghost: Power law behaviour in IR: $d_{0,IR}(p) \sim p^A$
- Gluon: Asymptotic freedom: $\omega_{0,UV}(p) \sim p$

$\Rightarrow$ Counterterms for ω_Λ and d_Λ^{-1} are needed:

- $\omega_\Lambda(p) = -\alpha + p$
- $d_\Lambda^{-1}(p) = const.$

$\Rightarrow$ How to choose the c.t. to satisfy the renormalisation conditions?

Solving Integral Equations

Integrate the differential flow equations:

$$\omega_q(p) - \omega_\Lambda(p) = \int_\Lambda^q dk \, \text{loop}(k,p)[d_k]$$

$$d_q^{-1}(p) - d_\Lambda^{-1}(p) = \int_\Lambda^q dk \, \text{loop}(k,p)[\omega_k, d_k]$$

$\Rightarrow$ Iteration of $\omega_q(p)$ and $d_q(p)$.

Fine Tuning - Gluon

$$p \sim \Lambda: \ \omega_0(p) - \omega_\Lambda(p) = \underbrace{\int_\Lambda^0 dk \, \text{loop}(k,p)[d_k]}_{\text{fit } \alpha+\beta p \text{ for } p\sim\Lambda}$$

With $\omega_\Lambda(p) = -\alpha + (1-\beta)p \Rightarrow \omega_0(p) = p$ for $p \sim \Lambda$

Fine Tuning - Ghost

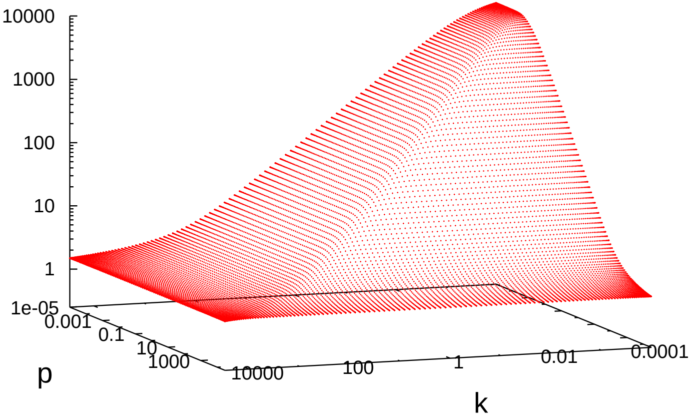
$$f(p) \propto p^A \Rightarrow p\frac{d}{dp}\ln f(p) = A$$

To obtain a power law, add d_Λ^{-1} to the integrated flow:

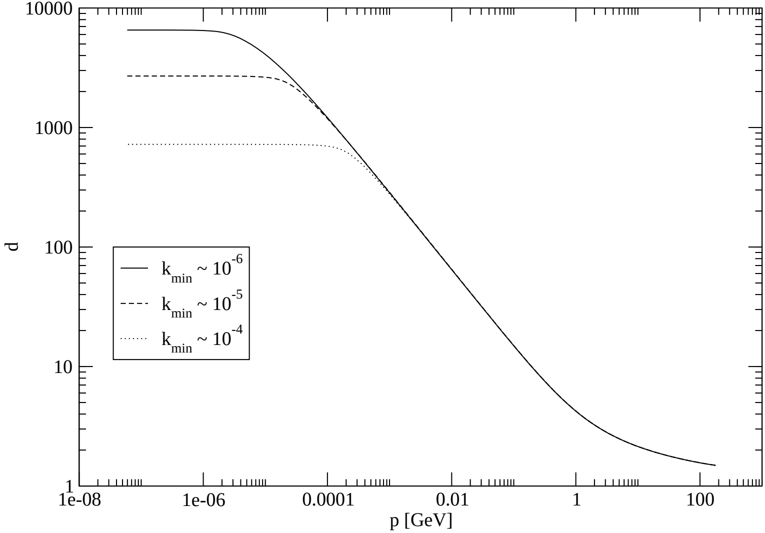
$$\frac{d}{dp}\left(p\frac{d}{dp}\ln\left(flow(p) + d_\Lambda^{-1}\right)\right) \stackrel{!}{=} 0$$

$\Rightarrow$ Determine d_Λ^{-1}.
$\Rightarrow$ No horizon condition has to be put in.

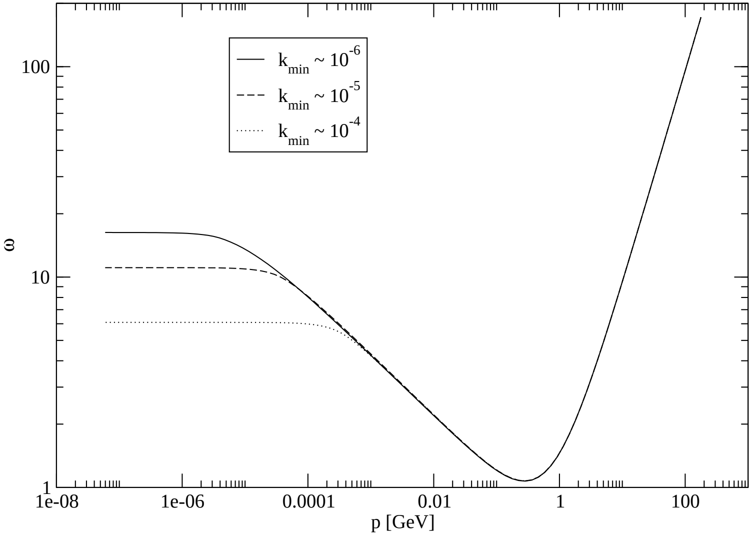
Full Flow of the Ghost Dressing Function $d_k(p)$



Ghost Flow


10000
1000
100
10
1
d
1e-08
1e-06
0.0001
0.01
1
100
p [GeV]
$k_{min} \sim 10^{-6}$
$k_{min} \sim 10^{-5}$
$k_{min} \sim 10^{-4}$

Gluon Flow



$k_{min} \sim 10^{-6}$
$k_{min} \sim 10^{-5}$
$k_{min} \sim 10^{-4}$
100
10
1
ω
1e-08
1e-06
0.0001
0.01
1
100
p [GeV]

Infrared Exponents

- Ghost

$$d_0(p)_{IR} \sim \ p^{-0.64}$$

- Gluon

$$\omega_0(p)_{IR} \sim \ p^{-0.28}$$

- These values of the exponents,
$\alpha_\omega = 0.28$, $\alpha_d = 0.64$,
satisfy the sum rule $\alpha_\omega = 2\alpha_d - 1$ obtained by infrared analysis of the DSE, see
W. Schleifenbaum, M. Leder, H. Reinhardt, PRD **73** :125019,2006

Comparison with Variational Results

Flow equation:

$\alpha_\omega = 0.28$, $\alpha_d = 0.64$

Weaker momentum dependence in the IR than in the variational solution:

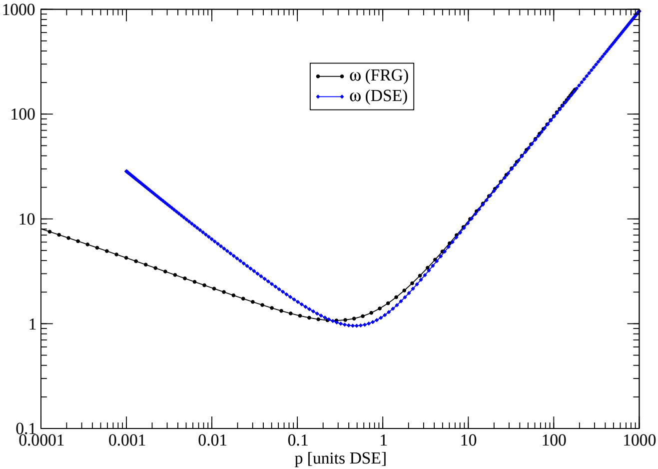
$\alpha_\omega = 0.6$, $\alpha_d = 0.8$,

C. Feuchter and H. Reinhardt, Phys. Rev. **D70** (2004) 105021 [arXiv:hep-th/0408236],

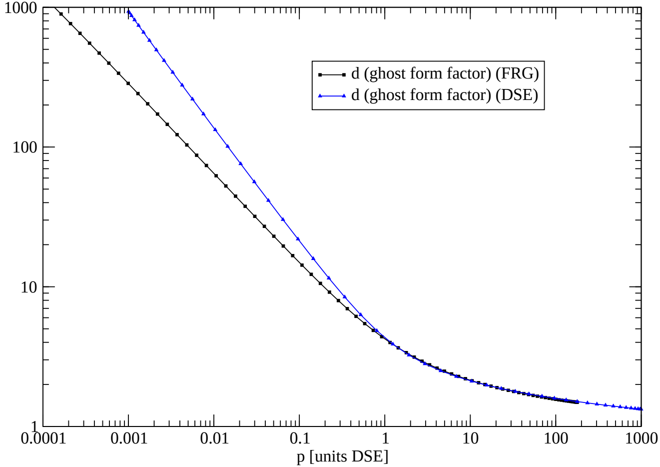
$\alpha_\omega = 1$, $\alpha_d = 1$,

D. Epple, H. Reinhardt and W. Schleifenbaum, Phys. Rev. **D75** (2007) 045011 [arXiv:hep-th/0612241].

Comparison with Variational Results - Gluon

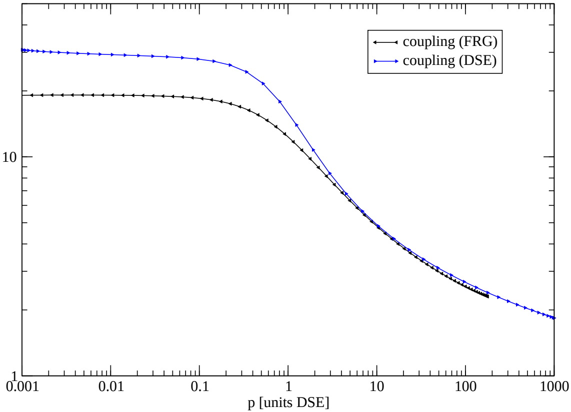


Comparison with Variational Results - Ghost



Comparison with Variational Results - Coupling

$$\alpha(p) \sim d^2(p)\omega^{-1}(p)p$$



Summary and Outlook

Summary

- FRG provides general framework for nonperturbative calculations.
- Ghost and gluon propagators have been calculated, using as input asymptotic freedom for the gluon and power law behaviour in the IR for the ghost.
- Quantitative behaviour of the propagators in the variational approach is largely reproduced.

Outlook

- Using optimised regulators $R_k(p)$ to minimise truncation effects
- Inclusion of dynamical quarks is straightforward

$Z_k \longrightarrow W_k \longrightarrow \Gamma_k$: The Average Effective Action Γ_k

$$Z[j] = e^{W[j]}$$

$$\Gamma[\phi] = -W[j] + j \cdot \phi$$

with j such that $\phi = \frac{\delta W[j]}{\delta j}$

$Z_k \longrightarrow W_k \longrightarrow \Gamma_k$: The Average Effective Action Γ_k

$$Z_k[j] = e^{W_k[j]}$$

$$\Gamma_k[\phi] = -W_k[j_k] + j_k \cdot \phi - \Delta S_k[\phi]$$

with j_k such that $\phi = \frac{\delta W_k[j_k]}{\delta j}$

Approximation schemes

Truncations should be systematic and consistent.

- Derivative expansion

$$\Gamma_k[\phi] = \int d^d x \left\{ U_k(\phi) + \frac{1}{2} Z_k(\phi)(\partial_\mu \phi)^2 + \mathcal{O}(\partial^4) \right\}$$

- Vertex expansion

$$\Gamma_k[\phi] = \sum_{n=0}^{\infty} \frac{1}{n!} \int d^d x_1 \dots d^d x_n \Gamma_k^{(n)}(x_1, \dots, x_n) \phi(x_1) \dots \phi(x_n)$$

Example: Propagator flow

$$\left.\frac{\delta^2}{\delta\phi\,\delta\phi}\right|_{\phi=0} \partial_t \Gamma_k[\phi] = \left.\frac{\delta^2}{\delta\phi\,\delta\phi}\right|_{\phi=0} \frac{1}{2} Tr \left[(\partial_t R_k) \left(\Gamma_k^{(2)}[\phi] + R_k \right)^{-1} \right]$$

Example: Propagator flow

$$\left.\frac{\delta^2}{\delta\phi\,\delta\phi}\right|_{\phi=0} \partial_t \Gamma_k[\phi] = \left.\frac{\delta^2}{\delta\phi\,\delta\phi}\right|_{\phi=0} \frac{1}{2} Tr \left[(\partial_t R_k) \left(\Gamma_k^{(2)}[\phi] + R_k \right)^{-1} \right]$$

$$\Rightarrow \partial_t \Gamma_k^{(2)} = Tr \left\{ (\partial_t R_k) [\Gamma_k^{(2)} + R_k]^{-1} \Gamma_k^{(3)} [\Gamma_k^{(2)} + R_k]^{-1} \Gamma_k^{(3)} [\Gamma_k^{(2)} + R_k]^{-1} \right\}$$
$$- \frac{1}{2} Tr \left\{ (\partial_t R_k) [\Gamma_k^{(2)} + R_k]^{-1} \Gamma_k^{(4)} [\Gamma_k^{(2)} + R_k]^{-1} \right\}$$

The flow equation for Γ_k

$$Z_k = e^{W_k} \; ; \; \Gamma_k = -W_k + J_k \cdot A + \bar{\sigma}_k \cdot c + \bar{c} \cdot \sigma_k - \frac{1}{2} A \cdot R_k \cdot A - \bar{c} \cdot \tilde{R}_k \cdot c$$

$$A = \frac{\delta W_k}{\delta j} \qquad c = \frac{\delta W_k}{\delta \bar{\sigma}} \qquad \bar{c} = -\frac{\delta W_k}{\delta \sigma}$$

The Functional Renormalisation Group Equation

$$\partial_t \Gamma_k = \frac{1}{2} Tr \left[(\partial_t R_k) \left(\frac{\delta^2 \Gamma_k}{\delta A \delta A} + R_k \right)^{-1} \right] - Tr \left[(\partial_t \tilde{R}_k) \left(\frac{\delta^2 \Gamma_k}{\delta c \delta \bar{c}} + \tilde{R}_k \right)^{-1} \right]$$

Truncation of the Theory Space

Restriction of theory space to

- gluon propagator
- ghost propagator

This would result in trivial flow equations.
$\Rightarrow$ Inclusion of one further operator:

- BARE ghost-gluon vertex

$\Rightarrow$ flow equations decoupled from the rest of theory space

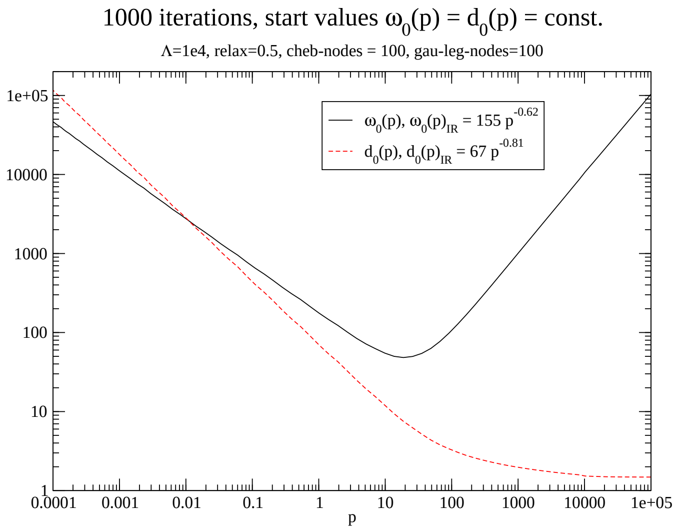
Approximation $\omega_k \longrightarrow \omega_0$, $d_k \longrightarrow d_0$ in the loop

- Ghost equation

$$d_0^{-1}(p) - d_\Lambda^{-1}(p) = N_c \left[\int \frac{d^d r}{(2\pi)^d} (2\omega_0(r) + R_k(r))^{-1} \cdot \right.$$
$$\left. \cdot \left(\frac{|\mathbf{r}+\mathbf{p}|^2}{d_0(|\mathbf{r}+\mathbf{p}|)} + \tilde{R}_k(|\mathbf{r}+\mathbf{p}|) \right)^{-1} \cdot \ldots \right]_{k=\Lambda}^{k=0}$$

$\Rightarrow$ Integrated RG equations correspond to DSE.

Results for Approximate Flow Equation



Results for Approximate Flow Equation

$$\omega_0(p)_{IR} \sim p^{-0.62} \quad d_0(p)_{IR} \sim p^{-0.81}$$

This result is close to one of two possible solutions found by IR-analysis of DSE and a numerical calculation resp. in

W. Schleifenbaum, M. Leder, H. Reinhardt, PRD **73** :125019,2006

C. Feuchter, H. Reinhardt, PRD **70**:105021,2004

Another possible solution

$$\omega_0(p)_{IR} \sim p^{-1} \quad d_0(p)_{IR} \sim p^{-1}$$

found by IR-analysis of DSE and a numerical calculation resp. in

W. Schleifenbaum, M. Leder, H. Reinhardt, PRD **73** :125019,2006

D. Epple, H. Reinhardt, W. Schleifenbaum, PRD **75**:045011,2007

is not seen here.

Flow Equation for ω_k

$$\partial_t \omega_k(p) = -\frac{N_c}{d-1} \int \frac{d^d r}{(2\pi)^d} \partial_t \tilde{R}_k(r) \left(\frac{r^2}{d_k(r)} + \tilde{R}_k(r) \right)^{-2} \cdot \\ \cdot \left(\frac{|\mathbf{r}+\mathbf{p}|^2}{d_k(|\mathbf{r}+\mathbf{p}|)} + \tilde{R}_k(|\mathbf{r}+\mathbf{p}|) \right)^{-1} \left(r^2 - (\hat{\mathbf{p}} \cdot \mathbf{r})^2 \right)$$

Flow Equation for d_k

$$
\begin{aligned}
\partial_t d_k^{-1}(p) = -N_c \int \frac{d^d r}{(2\pi)^d} \cdot \\
\cdot \Bigg[\partial_t R_k(r) [2\omega_k(r) + R_k(r)]^{-2} \left(\frac{|\mathbf{r}+\mathbf{p}|^2}{d_k(|\mathbf{r}+\mathbf{p}|)} + \tilde{R}_k(|\mathbf{r}+\mathbf{p}|) \right)^{-1} \cdot \ldots \\
+ \partial_t \tilde{R}_k(r) \left(\frac{r^2}{d_k(r)} + \tilde{R}_k(r) \right)^{-2} [2\omega_k(|\mathbf{r}+\mathbf{p}|) + R_k(|\mathbf{r}+\mathbf{p}|)]^{-1} \cdot \ldots \Bigg]
\end{aligned}
$$

Solving Differential Equations ?

- RG-equations are two coupled 1^{st} order ODE in k.
- Possible solution: Integrating numerically from initial conditions $[\omega_\Lambda(p), d_\Lambda(p)]$ down to $[\omega_0(p), d_0(p)]$.
- Initial conditions: $[\omega_\Lambda(p) = p, d_\Lambda(p) = 1]$

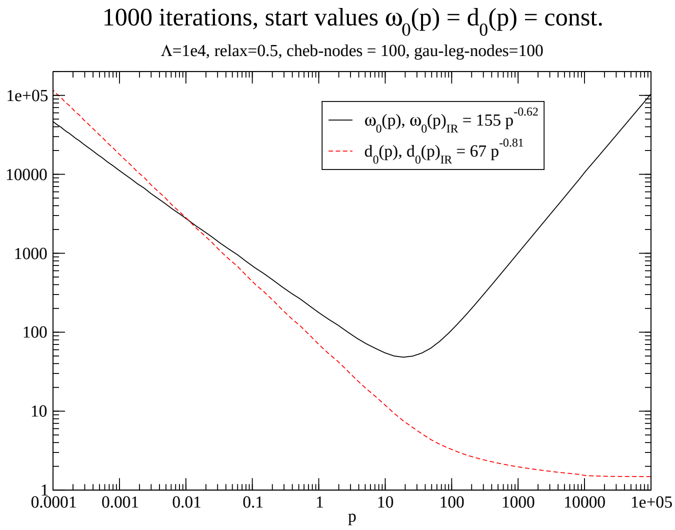
Approximation $\omega_k \longrightarrow \omega_0$, $d_k \longrightarrow d_0$ in the loop

$\Rightarrow$ Analytical integration of flow integrals feasible

- Gluon equation

$$\omega_0(p) - \omega_\Lambda(p) = \frac{N_c}{2(d-1)} \left[\int \frac{d^d r}{(2\pi)^d} \left(\frac{r^2}{d_0(r)} + \tilde{R}_k(r) \right)^{-1} \cdot \cdot \left(\frac{|\mathbf{r}+\mathbf{p}|^2}{d_0(|\mathbf{r}+\mathbf{p}|)} + \tilde{R}_k(|\mathbf{r}+\mathbf{p}|) \right)^{-1} \cdot \ldots \right]_{k=\Lambda}^{k=0}$$

Results for Approximate Flow Equation



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