Dyson equation for the Wilson loop

Markus Pak

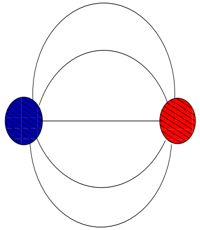
Institut für Theoretische Physik
Universität Tübingen

27.09.2009
Workshop des Graduiertenkollegs
"Hadronen im Vakuum, in Kernen und Sternen"

Confinement picture

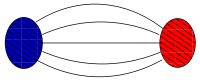
Quantum Electrodynamics

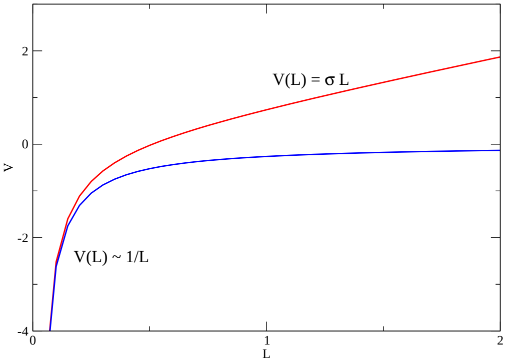
- Field lines spread out
- Abelian theory ⇒ photons do not self-interact



Quantum Chromodynamics

- Field lines concentrated in tube
- Non-abelian theory ⇒ gluons self-interact





QCD:

$$V(L) = C + \frac{A}{L} + \sigma L$$

QED:

$$V(L) = C + \frac{A}{L}$$

$\sigma \approx 1\mathrm{GeV/fm}$ (String tension)

Color confinement: "One cannot isolate a single quark."

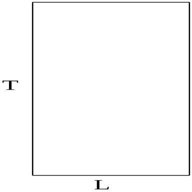
Table of Contents

Table of Contents

The Wilson loop

$$W_{\mathcal{C}}[A] \;=\; \frac{1}{N_C}\,\mathrm{Tr}\,\mathcal{P}\,\exp\left[-g\oint_{\mathcal{C}} dx^{\mu}\,A_{\mu}(x)\right]$$

- $\mathcal{P}$ = Path ordering, $A_\mu(x) = A_\mu^a(x)\,T^a$ at different space-time points do not commute



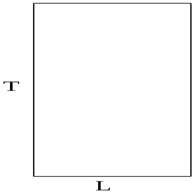
$$\langle W_{\mathcal{C}}[A]\rangle \;\underset{T\gg L}{\longrightarrow}\; e^{-V(L)\,T}$$

Expectation value:

$$\langle W_{\mathcal{C}}[A]\rangle \;=\; \langle\Psi_0|W_{\mathcal{C}}[A]|\Psi_0\rangle \;=\; \int \mathcal{D}A\,W_{\mathcal{C}}[A]\,e^{-S_G[A]}$$

$$W_{\mathcal{C}}[A] \;=\; \frac{1}{N_C}\,\mathrm{Tr}\,\mathcal{P}\,\exp\left[-g\oint_{\mathcal{C}} dx^{\mu}\,A_{\mu}(x)\right]$$

- $\mathcal{P}$ = Path ordering, $A_\mu(x) = A^a_\mu(x)\,T^a$ at different space-time points do not commute



$$\langle W_{\mathcal{C}}[A]\rangle \;\xrightarrow[T\gg L]{}\; e^{-V(L)\,T}$$

Expectation value:

$$\langle W_{\mathcal{C}}[A]\rangle \;=\; \langle\Psi_0|W_{\mathcal{C}}[A]|\Psi_0\rangle \;=\; \int \mathcal{D}A\,W_{\mathcal{C}}[A]\,e^{-S_G[A]}$$

Problem: Path ordering in the continuum!

Table of Contents

Weak coupling expansion

- We expand $\langle W_{\mathcal{C}}[A] \rangle$ to $\mathcal{O}(g^2) \Rightarrow$ path ordering can be ignored
- Gluon action in perturbation theory

$$S_G[A] = \frac{1}{2} \int d^4x \, d^4y \, A^\mu(x) \, D^{-1}_{\mu\nu}(x,y) \, A^\nu(y)$$

- Line element dx_μ replaced by source J
- Integral over a quadratic form

$$\langle W_{\mathcal{C}}[A] \rangle = \int \mathcal{D}A \exp\left[-AD^{-1}A - JA \right]$$

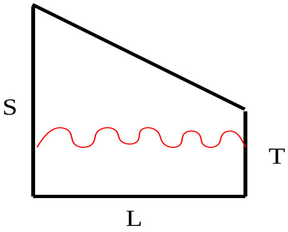
- Gauge has to be fixed!

$$\langle W_{\mathcal{C}}[A] \rangle = \exp\left[-\frac{g^2}{2} \, C_2 \oint_{\mathcal{C}} dx^\mu \oint_{\mathcal{C}} dy^\nu \, D_{\mu\nu}(x,y) \right]$$

Assumptions

Consider a trapezoidal Wilson loop:

- Loop integral has 16 contributions
- Temporal paths are dominant for $S, T >> L$
- Take only two temporal paths into account

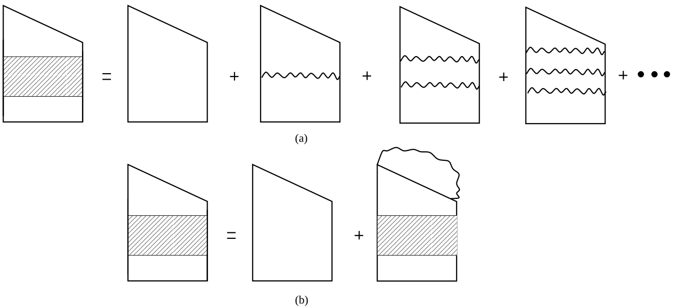


$$\langle W_{\mathcal{C}}[A] \rangle = W(S,T;L) = 1 + g^2\, C_2 \int_0^S ds \int_0^T dt\, D\left((x(s)-x(t))^2\right)$$

Projected gluon propagator:

$$D\left((x(s)-x(t))^2\right) = \dot{x}^{\mu}(s)\, D_{\mu\nu}\left((x(s)-x(t))^2\right)\, \dot{x}^{\nu}(t)$$

Resum all diagrams with gluon lines connecting the temporal paths



Dyson equation:

$$W(S,T;L) \,=\, 1 + g^2\, C_2 \int_0^S ds \int_0^T dt\, D((x(s)-x(t))^2)\, W(s,t;L)$$

References

SUSY Yang-Mills:

- J.K. Erickson, G.W Semenoff and K. Zarembo
  Wilson loops in N=4 supersymmetric Yang-Mills theory
  Nucl.Phys. B **582**, 155 (2000) [arXiv:hep-th/0003055]

- J.K. Erickson, G.W Semenoff, R.J. Szabo and K. Zarembo
  Static potential in N = 4 supersymmetric Yang-Mills theory
  Phys. Rev. D **61**, 105006 (2000) [arXiv:hep-th/9911088]

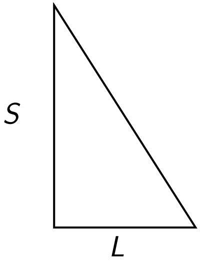
Non-SUSY Yang-Mills:

- A.V Zayakin and J. Rafelski
  The confinement property in $SU(3)$ gauge theory
  Phys. Rev.D **80**, 034024 (2009) [arXiv:hep-ph/0905.2317]
- H. Reinhardt, M. Pak
  in preparation

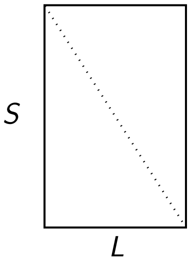
Limitations

I) Wrong boundary condition

$$W(S, T=0; L) = 1 \quad W(S=0, T; L) = 1$$



$T = 0$



$T = S$

Assume area law:

Rectangular loop: $\langle W_{\mathcal{C}}[A] \rangle \sim \exp\left[-\sigma S L\right]$

Triangle loop: $\langle W_{\mathcal{C}}[A] \rangle \sim \exp\left[-\sigma S L/2\right] \neq 1$

Limitations

II) Includes only one pair of paths

- Applies only to strongly asymmetric loops
- Only for intermediate distances $L \ll T$
- Limit $L \to \infty$ not accessible
- Result **not** gauge-invariant

Evaluation of the static potential

Idea: Reduce integral equation to Schrödinger equation

- Differentiate with respect to S and T

$$\frac{\partial^2 W(S,T;L)}{\partial S \partial T} = g^2 C_2 D\left(L^2 + (S-T)^2\right) W(S,T;L)$$

- Separate the variables

$$r = \frac{S-T}{L}, \quad R = \frac{S+T}{L}, \quad \frac{\partial^2}{\partial S \partial T} = \frac{1}{L^2}\left(\frac{\partial^2}{\partial R^2} - \frac{\partial^2}{\partial r^2}\right)$$

- Choose ansatz

$$W(r;R) = \sum_n \varphi_n(r)\left[c_n^+ e^{\frac{\Omega_n}{2}R} + c_n^- e^{-\frac{\Omega_n}{2}R}\right]$$

Schrödinger equation

$$\left[-\frac{d^2}{dr^2} + U(r)\right] \varphi_n(r) = -\frac{\Omega_n^2}{4}\,\varphi_n(r)$$

$$U(r) = -g^2 C_2\, L^2\, D(L^2(1+r^2))$$

Static quark potential

$$V(L) = -\lim_{T\to\infty} \frac{1}{T} \ln W(S = T, T; L) = -\frac{\Omega_0(L)}{L}$$

Schrödinger equation

$$\left[-\frac{d^2}{dr^2} + U(r)\right] \varphi_n(r) = -\frac{\Omega_n^2}{4} \varphi_n(r)$$

$$U(r) = -g^2 C_2 L^2 D(L^2(1+r^2))$$

Static quark potential

$$V(L) = -\lim_{T\to\infty} \frac{1}{T} \ln W(S=T, T; L) = -\frac{\Omega_0(L)}{L}$$

$\Rightarrow$ Compute the lowest eigenvalue

Table of Contents

Temporal gluon propagator

$$g^2 D_{00}(x-y) = \underbrace{V_C(|\mathbf{x}-\mathbf{y}|)\,\delta(x^0-y^0)}_{\sim\text{confinement}} + P(x-y)$$

Color-Coulomb potential $V_C(L) = C + \frac{A}{L} + \sigma_C L$

Dyson-equation for confining part:

$$W(T;L) = 1 - C_2 V_C(L) \int_0^T dt\, W(t;L)$$

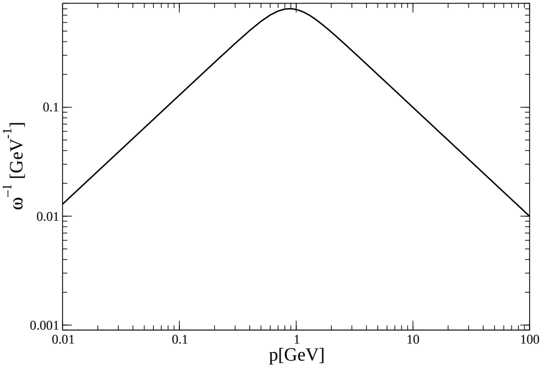
$$\Updownarrow$$

$$\frac{dW(T;L)}{dT} = -C_2 V_C(L) W(T;L) \qquad \text{boundary: } W(T=0;L) = 1$$

Result: $W(T;L) = e^{-C_2 V_C(L) T} \Rightarrow$ Area law!

Spatial static gluon propagator

Apply Dyson equation to spatial Wilson loop



[C. Feuchter and H. Reinhardt, PRD**70**, 105021 (2004)]

$$D_{ij}(\mathbf{p}) = \frac{1}{2}\, t_{ij}(\mathbf{p})\, \omega^{-1}(p)$$

Transversal projector:

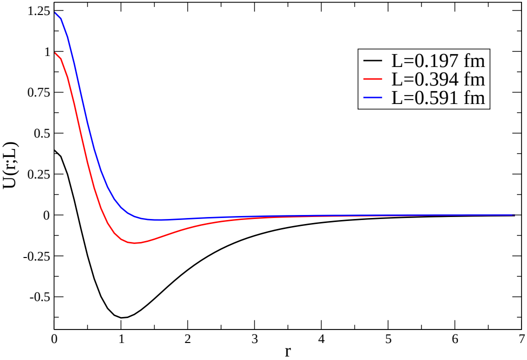
$$t_{ij}(\mathbf{p}) = \delta_{ij} - \frac{p_i\, p_j}{p^2}$$

Gribov fit:

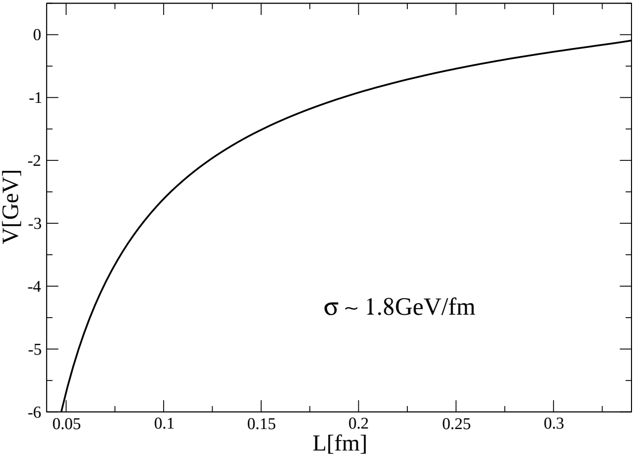
$$\omega(p) = \sqrt{p^2 + \frac{M^4}{p^2}}$$

Schrödinger potential

$$U(r) = -g^2 C_2 \, L^2 \, D(L^2(1 + r^2)) \, , \qquad g^2(0) = \frac{32\pi^2}{3}$$

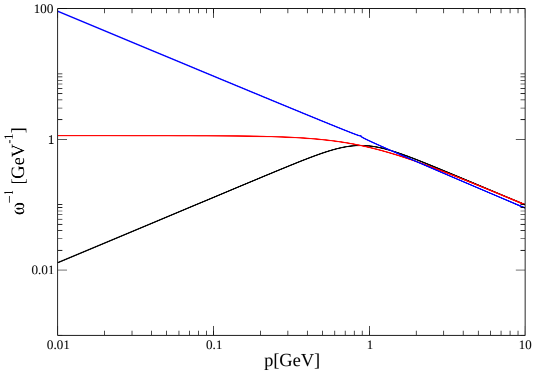


Static quark potential



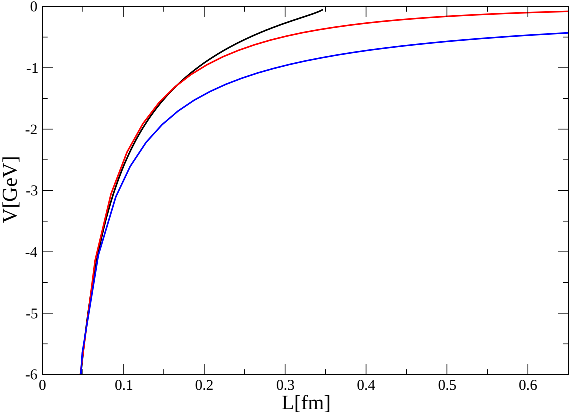
Both Coulombic and confining region observed!

Other propagators



- Scaling propagator
  $\omega(p) = \sqrt{p^2 + \frac{M^4}{p^2}}$
- Decoupling propagator
  $\omega(p) = \sqrt{p^2 + M^2}$
- Tree-level propagator
  $\omega(p) = \sqrt{p^2}$

Static quark potential



Scaling propagator ⟶ string tension

Decoupling, tree-level ⟶ no string tension

Summary and conclusion

Wilson loop in a continuum formulation:

- From a weak coupling expansion
- Resummation of diagrams

Despite the approximations

- Coulomb gauge gluon propagator: coulombic and confining region
- Tree-level propagator: only coulombic region

Thank you for your attention!