Electromagnetic Structure of Hadrons in Point-Form Relativistic Quantum Mechanics

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Structure of matter: nuclei

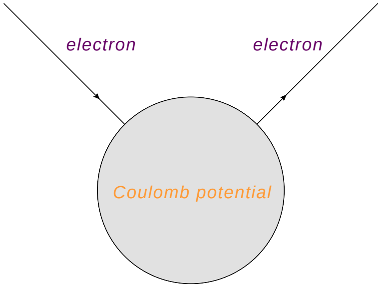
- Geiger, Marsden 1909: α particles reflected by metal foil
- Rutherford: many α particles reflected under a big scattering angle $\Rightarrow$ small, massive, positively charged nucleus $\Rightarrow$ Rutherford formula for angular distribution of α particles $\propto \frac{1}{\sin^4(\phi/2)}$
- Hofstadter et al. 1950's at SLAC: electron-gold elastic scattering $\Rightarrow$ deviations from Rutherford formula (for charged pointlike particle scattering)
  angular distribution of electrons falls *below* the "point-nucleus" prediction $\Rightarrow$ **form factor**: characteristic of spatial extentions of charge density
  $\Rightarrow$ nucleus has finite spatial extension $\Rightarrow$ consists of constituents: nucleons

Structure of matter: nucleons

- Hofstadter et al. 1963: elastic scattering of electrons from nucleons ⇒ proton has well-defined form factor⇒ nucleon is not structureless
- inelastic electron scattering: nucleon spectroscopy (internal motion of constituents)
  similar for mesons: sequence of excited states
  ⇒ nucleons and mesons consist of constituents
- Gell-Mann and Zweig 1964: nucleon-like states (baryons) made out of 3 and pion-like (mesons) made out of 2 spin-1/2 constituents: quarks
- Stanford 1969: deep inelastic electron-nucleon scattering experiments ⇒ existence of (pointlike) quarks ⇒ nucleon as bound state of constituents

Coulomb scattering of an electron

- electron (charged spin-1/2 particle): $e^-(p_e, \sigma_e) \to e^-(p'_e, \sigma'_e)$



- time dependent perturbation theory $\Rightarrow$ first order transition amplitude

$$\mathcal{A}^{(1)}_{e^-} = -\mathrm{i} \int \mathrm{d}^4 x \underbrace{j^\mu_{e^-}(x)}_{-e\bar{u}_{\sigma'_e}(p'_e)\gamma^\mu u_{\sigma_e}(p_e)\mathrm{e}^{-i(p_e - p'_e)\cdot x}} A_\mu(x)$$

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Pointlike charge distribution

- pure Coulomb potential $A^\mu \propto \left(\frac{e}{|\mathbf{x}|}, \mathbf{0}\right)^\mu$
  (solution of Poisson equation $\nabla^2 A^0 = -e\rho(\mathbf{x}) = -e\delta^3(\mathbf{x})$)
  ρ charge density distribution $\int \mathrm{d}^3x\rho(\mathbf{x}) = 1$
  $\Rightarrow$ amplitude $\mathcal{A}^{(1)}_{e^-} \propto \delta\left(p_e'^0 - p_e^0\right) \frac{1}{|\mathbf{q}|^2} e^2 \bar{u}_{\sigma_e'} \gamma^0 (p_e') u_{\sigma_e}(p_e)$
  momentum transfer $\mathbf{q} = \mathbf{p}_e' - \mathbf{p}_e$
- calculate cross section (Mott cross section $\equiv$ relativistic Rutherford formula for spin-1/2 particles)

$$\left(\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega}\right)_{\mathrm{Mott}} = \frac{\alpha^2 (p_e^0)^2 \left(1 - \frac{|\mathbf{p}_e|^2}{(p_e^0)^2}\sin^2(\theta/2)\right)}{4|\mathbf{p}_e|^4 \sin^4(\theta/2)} \xrightarrow{\frac{|\mathbf{p}_e|^2}{(p_e^0)^2} \to 1} \frac{\alpha^2 (p_e^0)^2 \cos^2(\theta/2)}{4|\mathbf{p}_e|^2 \sin^4(\theta/2)}$$

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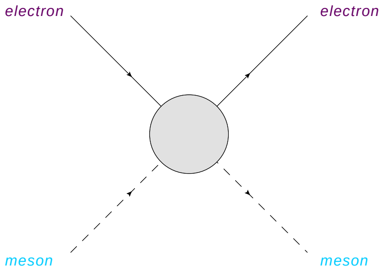
Extended (non-pointlike) charge distribution: form factor

- $A^0(\mathbf{x})$ Coulomb potential of spread out charge density $\rho(\mathbf{x})$
  (solution of Poisson equation $\nabla^2 A^0(\mathbf{x}) = -e\rho(\mathbf{x})$)
  $\Rightarrow$ in amplitude for transition $\mathcal{A}^{(1)}_{e^-}$ replace
  $e\int \mathrm{d}^3x \frac{\mathrm{e}^{-i\mathbf{q}\cdot\mathbf{x}}}{|\mathbf{x}|} = \frac{e}{\mathbf{q}^2} \rightarrow \frac{e}{\mathbf{q}^2}\int \mathrm{d}^3x \mathrm{e}^{-i\mathbf{q}\cdot\mathbf{x}}\rho(\mathbf{x}) = \frac{e}{\mathbf{q}^2}F(\mathbf{q})$
  $F(\mathbf{q})$ **static form factor** describes the structure of a non-pointlike charge distribution
  $\Rightarrow$ transition amplitude $\mathcal{A}^{(1)}_{e^-} \propto \delta\left(p'^0_e - p^0_e\right) \frac{F(\mathbf{q})}{\mathbf{q}^2} e^2 \bar{u}_{\sigma'_e}\gamma^0\left(p'_e\right) u_{\sigma_e}\left(p_e\right)$
- $F(\mathbf{q})$ decreases smoothly away from $q^2 = 0$
  normalization $F(\mathbf{0}) = 1$ (charge)
- cross section $\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} \propto |\mathcal{A}_{e^-}|^2 \Rightarrow$ cross section at higher $\mathbf{q}^2$ will drop below the Mott cross section (pointlike) value

Pointlike charged spin-0 particle

- Lorentz-invariant generalization of the static case: quantum field theory
- scattering of an electron at a pointlike, charged spin-0 particle s^+

$$e^-(p_e, \sigma_e) + s^+(p_s) \to e^-(p'_e, \sigma'_e) + s^+(p'_s)$$

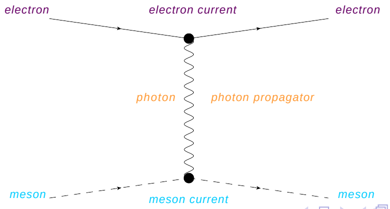


Scattering Operator

- e^- and s^+ charged particle: interaction by exchange of a virtual particle (virtual photon γ^*)
- interaction Lagrangians ($s\gamma$- and $e\gamma$-vertices)
  $$\hat{\mathcal{L}}^{s\gamma}_{\text{int}}(x) = \underbrace{\mathrm{i}e\hat{\phi}^\dagger(x)\overleftrightarrow{\partial}^\mu\hat{\phi}(x)}_{=:\hat{j}^\mu_s(x)}\hat{A}_\mu(x)$$ and
  $$\hat{\mathcal{L}}^{e\gamma}_{\text{int}}(x) = \underbrace{-e\hat{\bar{\psi}}(x)\gamma^\mu\hat{\psi}(x)}_{=:\hat{j}^\mu_e(x)}\hat{A}_\mu(x)$$
- insert interaction Lagrangian in scattering operator $\hat{S} = \mathcal{T}\exp\left(-\mathrm{i}\int \mathrm{d}^4x\hat{\mathcal{L}}_{\text{int}}(x)\right) \Rightarrow$ Dyson expansion of the S-operator

Scattering Amplitude

- consider matrix element: $\left\langle p_e', \sigma_e'; p_s' | \hat{S} | p_e, \sigma_e; p_s \right\rangle$
- first non-vanishing scattering amplitude $\Rightarrow$ second order:
  $\mathcal{A}^{(2)}_{e^- s^+} \propto \delta^4 \left(p_e + p_s - p_e' - p_s'\right) \underbrace{j^\mu_{s^+} \frac{g_{\mu\nu}}{q^2} j^\nu_{e^+}}_{\propto \mathcal{M}_{s^+ e^-}}$ where $j^\mu_{s^+} := e(p_s + p_s')^\mu$ and
  $j^\mu_{e^+} := -e\bar{u}_{\sigma_e'}\left(p_e'\right)\gamma^\mu u_{\sigma_e}\left(p_e\right)$ are the transition currents
  $\frac{g_{\mu\nu}}{q^2}$ photon propagator
  $\Rightarrow \mathcal{M}_{s^+ e^-}$ (gauge) invariant amplitude: 2 currents hooked together by photon propagator



No structure cross section

- calculate "no structure" cross section in laboratory frame

$$\left(\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega}\right)_{\text{no structure}} = \frac{\alpha^2 \cos^2(\theta/2)\overbrace{\left(1 + \frac{2|\mathbf{p}_e|}{m_s}\sin^2(\theta/2)\right)}^{\frac{|\mathbf{p}_e|}{|\mathbf{p}_e'|}\text{(recoil)}}}{4|\mathbf{p}_e|^2 \sin^4(\theta/2)}$$

- consider limit $m_s \to \infty$ (no recoil) $\Rightarrow$ recover Mott cross section in relativistic limit: $\frac{\alpha^2 \cos^2(\theta/2)}{4|\mathbf{p}_e|^2 \sin^4(\theta/2)}$
  $\Rightarrow$ scaling property
- experiment: observe scaling $\Leftrightarrow$ pointlike particles

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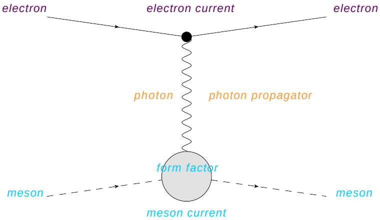
Non-pointlike particle: pion form factor

- charged meson: composite system of quark-antiquark, e.g. pion π^+: $u\ \bar{d}+\ldots$
- quarks
  1. interact locally with γ^*
  2. interact strongly via interactions of quantum chromodynamics (QCD)

  $\Rightarrow \pi^+$ cannot have pointlike interactions with the electromagnetic field
- bound state: strong interactions cannot to be treated peturbatively $\Rightarrow$ no full understanding of electromagnetic structure of hadrons
- phenomenological quantity: pion form factor
  describes non-pointlike aspect of hadronic structure of π^+
  Lorentz invariant generalization of static form factor

Electron-pion scattering

- process $e^-(p_e, \sigma_e) + \pi^+(p_s) \to e^-(p_e', \sigma_e') + \pi^+(p_s')$
- lowest order perturbation theory: one-photon exchange approximation



- 2 requirements:
  1. Lorentz covariance
  2. current conservation

Lorentz covariance and current conservation

- generalize pointlike $s^+ s^+ \gamma$ vertex $j^\mu_{s^+} \rightarrow j^\mu_{\pi^+}$ which includes strong interaction effects between the quarks which cannot destroy four-vector character of the current
- Lorentz covariance $\Rightarrow$ most general electromagnetic vertex of a pion $j^\mu_{\pi^+} = e\left[F\left(q^2\right)(p_s + p_s')^\mu + G\left(q^2\right)q^\mu\right]$
  form factors F and G
- current conservation condition
  $q_\mu j^\mu_{\pi^+} \stackrel{!}{=} 0 \Rightarrow G(q^2) = 0, \forall q^2 \neq 0$

Pion form factor

- all strong virtual interaction effects at $\pi^+\pi^+\gamma$ vertex described by one scalar function of the virtual photons four momentum: electromagnetic form factor of the pion $F\left(q^2\right)$
- electric charge $+e$ of pion $\equiv$ coupling at zero momentum transfer $\Rightarrow$ normalization: $F(0) = 1$
- effect of pion structure in invariant amplitude: replace
  $j^\mu_{s^+} \rightarrow j^\mu_{\pi^+} = F\left(q^2\right) j^\mu_{s^+} \Rightarrow$
  $\mathcal{M}_{\pi^+ e^-} \propto F\left(q^2\right) j^\mu_{s^+} \frac{g_{\mu\nu}}{q^2} j^\nu_{e^-}$
  $\Rightarrow \left(\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega}\right)_{\text{pion structure}} = F^2\left(q^2\right)\left(\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega}\right)_{\text{no structure}}$
- since $q^2 < 0 \Rightarrow F(q^2)$ probed for negative values of
  $q^2 =: -Q^2$
  $F(Q^2)$ falls off as Q^2 increases $\Rightarrow$ experiment: cross section below pointlike value

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Point-form relativistic quantum mechanics

- Relativistic quantum mechanics (RQM): effective model with correct relativistic treatment
- point form of dynamics: $\hat{P}^{\mu}$ dynamical, $\hat{M}^{\mu\nu}$ kinematical
  $\Rightarrow$ manifest Lorentz-covariant formulation

  Dirac, 1949
- find a representation of generators $\left\{\hat{P}^{\mu}, \hat{M}^{\mu\nu}\right\}$ on $\mathcal{H}$ that satisfy Poincaré algebra
  1. free theory: easy to achieve
  2. interacting theory: non-linear constraints on interaction terms that are hard to satisfy

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Bakamjian-Thomas Construction

- Bakamjian-Thomas construction: interactions included in mass operator

  Bakamjian, Thomas 1953

  point-form RQM: $\hat{P}^{\mu}_{\text{free}} + \hat{P}^{\mu}_{\text{int}} = \left(\hat{M}_{\text{free}} + \hat{M}_{\text{int}}\right) \hat{V}^{\mu}_{\text{free}}$

  $\Rightarrow \left\{\hat{M}, \hat{\mathbf{V}}_{\text{free}}, \hat{M}^{\mu\nu}_{\text{free}}\right\}$ satisfy certain commutation relations $\Rightarrow$ linear constraints on interaction terms

- multiparticle states, basis: velocity states

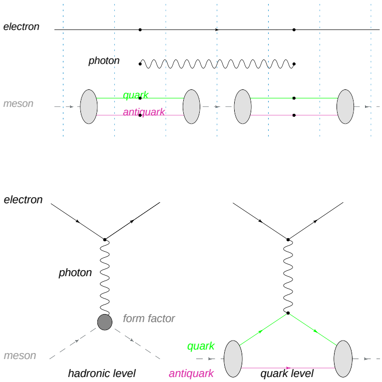
  Klink 1998

  $|v; \mathbf{k}_1, \mu_1; \mathbf{k}_2, \mu_2; \ldots ; \mathbf{k}_n, \mu_n\rangle = \hat{U}\left(B_c\left(v\right)\right) |\mathbf{k}_1, \mu_1; \mathbf{k}_2, \mu_2; \ldots ; \mathbf{k}_n, \mu_n\rangle$

Meson form factor

- calculate electromagnetic form factor of pion in Bakamjian-Thomas-type of approach
- usual attempts: ansatz for electromagnetic current
- our approach: derive current with all required properties
- elastic electron-meson scattering in 1-photon approximation

Elastic electron-meson scattering



calculate optical potential on hadronic and quark level ⇒ extract form factor

2-channel system

Hadronic level

- mass eigenstate $|\Psi\rangle = \begin{pmatrix} |\Psi_{es}\rangle \\ |\Psi_{es\gamma}\rangle \end{pmatrix}$
- coupled-channel mass operator $\hat{M} = \begin{pmatrix} \hat{M}_{es} & \hat{K}^\dagger \\ \hat{K} & \hat{M}_{es\gamma} \end{pmatrix}$
- mass eigenvalue equation
  $\hat{M}|\Psi\rangle = m|\Psi\rangle \;\Rightarrow$ system of coupled equations
- solve for $|\Psi_{es\gamma}\rangle \Rightarrow$ non-linear eigenvalue equation

$$\underbrace{\hat{K}^\dagger \left(\hat{M}_{es\gamma} + m\right)^{-1} \hat{K}}_{=\hat{V}_{\mathrm{opt}}(m)} |\Psi_{es}\rangle = \left(\hat{M}_{es} + m\right) |\Psi_{es}\rangle .$$

Optical potential

Hadronic level

- electromagnetic vertex interaction

  Klink 2003

  $$\left\langle v'; \mathbf{k}'_e, \mu'_e; \mathbf{k}'_s; \mathbf{k}_\gamma, \mu_\gamma \left| \hat{K} \right| v; \mathbf{k}_e, \mu_e; \mathbf{k}_s \right\rangle \propto v^0 \delta^3 \left(\mathbf{v} - \mathbf{v}' \right)$$
  $$\times \left\langle v; \mathbf{k}'_e, \mu'_e; \mathbf{k}'_s; \mathbf{k}_\gamma, \mu_\gamma \left| \left(F\left(\Delta m\right) \hat{\mathcal{L}}^{s\gamma}_{\text{int}}(0) + \hat{\mathcal{L}}^{e\gamma}_{\text{int}}(0) \right) \right| v; \mathbf{k}_e, \mu_e; \mathbf{k}_s \right\rangle$$

  assumption: total-velocity conservation at electromagnetic vertices (approximation)

- matrix element of optical potential

  $$\left\langle v'; \mathbf{k}'_e, \mu'_e; \mathbf{k}'_s \left| \hat{V}_{\text{opt}}\left(m\right) \right| v; \mathbf{k}_e, \mu_e; \mathbf{k}_s \right\rangle \propto$$
  $$v^0 \delta^3 \left(\mathbf{v} - \mathbf{v}' \right) F\left(Q^2, \ldots\right) j^\mu_{s^+} \frac{g_{\mu\nu}}{q^2} j^\nu_{e^-}$$

Optical potential

Quark level

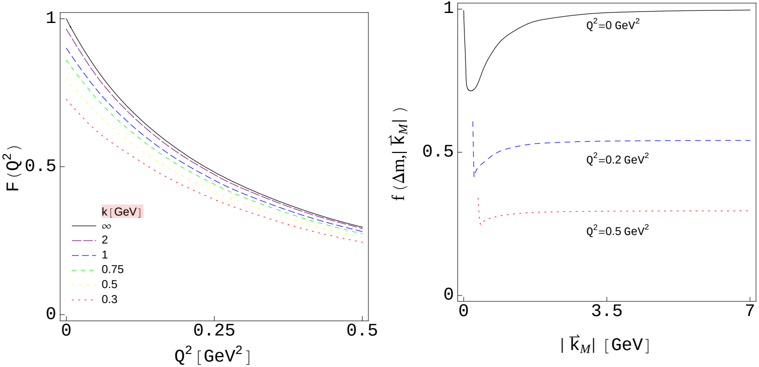
- confinement: instantaneous potential
  $\hat{M}_{es} \to \hat{M}_{e\pi} = \hat{M}_{eq\bar{q}} + \hat{V}_{\text{conf}}$
  $\hat{M}_{es\gamma} \to \hat{M}_{e\pi\gamma} = \hat{M}_{eq\bar{q}\gamma} + \hat{V}_{\text{conf}}$
- matrix element of optical potential
  $\left\langle \underline{v}'; \underline{\mathbf{k}}'_e, \underline{\mu}'_e; \underline{\mathbf{k}}'_s, n', \tilde{m}'_j, j' \middle| \hat{V}_{\text{opt}}(m) \middle| \underline{v}; \underline{\mathbf{k}}_e, \underline{\mu}_e; \underline{\mathbf{k}}_s, n, \tilde{m}_j, j \right\rangle \propto$
  $\underline{v}^0 \delta^3 (\underline{\mathbf{v}} - \underline{\mathbf{v}}') \, j^\mu_{\pi^+} \frac{g_{\mu\nu}}{q^2} j^\nu_{e^-}$
  microscopic meson current $j^\mu_{\pi^+} \propto \sum \int \cdots \Psi^* D^{\frac{1}{2}}(R_W) j^\mu_q \Psi$

Form factor

- identify form factor by comparing matrix elements of optical potential on hadronic and quark level
  $F\left(Q^2, \ldots\right) j^{\mu}_{s^+} j_{e^- \mu} = j^{\mu}_{\pi^+} j_{e^- \mu}$ $\Rightarrow$ can depend also on other invariants, like e.g. total invariant mass of system
- simple model: $\Psi \propto \exp\left(-\frac{\tilde{\mathbf{k}}_q^2}{2a^2}\right)$ harmonic oscillator wave function
  $\Rightarrow$ 2 parameters: m_q, a

Form factor

$|\mathbf{k}_s|$-dependence



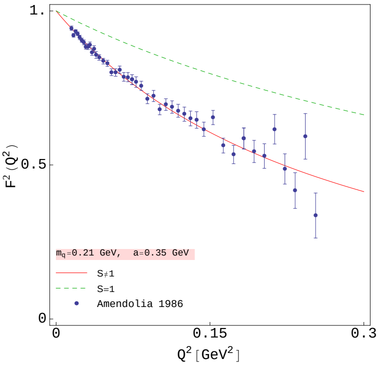
⇒ $|\mathbf{k}_s|$-dependence (⇐ approximation: velocity conservation at vertex) ⇒ vanishes fast for increasing $|\mathbf{k}_s|$ (>2 GeV)

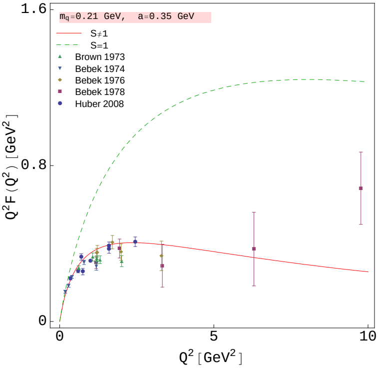
Infinite momentum frame

$|\mathbf{k}_s| \to \infty$

- extract form factor where $|\mathbf{k}_s|$-dependence vanishes: $|\mathbf{k}_s| \to \infty$
  $\Rightarrow j^{\mu}_{\pi^+} \longrightarrow F\left(Q^2\right) j^{\mu}_{s^+} \Rightarrow$ electromagnetic pion form factor: overlap integral
  $F\left(Q^2\right) = \int \mathrm{d}^3 \tilde{k}'_q \sqrt{\frac{m_{q\bar{q}}}{m'_{q\bar{q}}}} \mathcal{S} \Psi^* \left(\tilde{\mathbf{k}}'_q\right) \Psi\left(\tilde{\mathbf{k}}_q\right)$
  B., Schweiger, Fuchsberger, Klink 2008
- $|\mathbf{k}_s| \to \infty$ means that subprocess $\gamma^* \pi^+ \to \pi^+$ in infinite momentum frame is considered
  $\Rightarrow$ equivalence with front form result: overlap integrals connected by variable transformation $\left\{\tilde{\mathbf{k}}'_q\right\} \to \{\mathbf{k}_\perp, x\}$
  Chung, Coester, Polyzou 1988

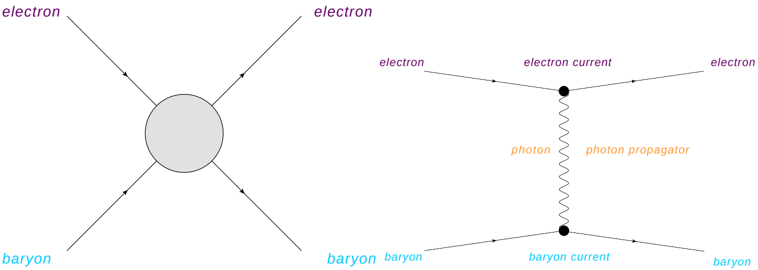
Comparison with experiment





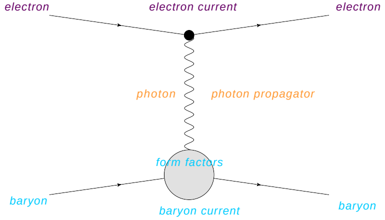
Electron-baryon scattering

- electron-proton scattering process
  $e^-(p_e, \sigma_e) + p\,(p_p, \sigma_p) \rightarrow e^-(p'_e, \sigma'_e) + p\,(p'_p, \sigma'_p)$



Nucleon Form Factors

- $pp\gamma$ vertex not known



- most general form of proton electromagnetic current satisfying Lorentz covariance and current conservation

$$j_p^\mu = e\bar{u}_{\sigma'_p}(p'_p)\left[F_1(q^2)\gamma^\mu + i\frac{\kappa F_2(q^2)}{2m_p}\sigma^{\mu\nu}q_\nu\right]u_{\sigma_p}(p_p)$$

F_1 Dirac charge form factor , F_2 Pauli anomalous magnetic moment form factor where $F_1(0) = 1$ (proton charge) and $F_2(0) = 1$ (anomalous magnetic moment/κ)

Rosenbluth cross section

- electron-proton scattering cross section (Rosenbluth cross section) $\left(\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega}\right)_{\text{Rosenbluth}} =$
  $\left(\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega}\right)_{\text{no structure}} \left[\left(F_1^2 - \frac{\kappa^2 q^2}{4m_p^2} F_2^2\right) - \frac{q^2(F_1+\kappa F_2)^2 \tan^2\left(\frac{\phi}{2}\right)}{2m_p^2}\right]$
  $\left(\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega}\right)_{\text{Rosenbluth}} \xrightarrow{F_1=1, F_2=0} \left(\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega}\right)_{\mu^- \text{ no structure}}$
- define electric and magnetic Sachs form factors
  $G_E := F_1 + \frac{\kappa q^2}{4m_p^2} F_2$ and $G_M := F_1 + \kappa F_2$
  G_E and G_M decrease as Q^2 increases

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Summary

- structure of a charge distribution: form factor
  non-relativistically: Fourier transform of charge density
  Lorentz invariant generalization: form factor takes into account (in relativistically invariant manner) the non-pointlike aspect of hadronic structure
- point form: Poincaré invariant coupled-channel approach to electron-meson scattering
- assumption: total velocity conservation at interaction vertices (approximation which satisfies Poincaré invariance) $\Rightarrow$ electromagnetic meson form factor: dependence on Q^2, invariant mass
- form factor extracted in infinite momentum frame ($s \longrightarrow \infty$) $\Rightarrow$ equivalence to front form calculations
- outlook: extension to heavy-light systems, baryons ...