

Introduction to Chiral Perturbation Theory

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I. Standard Model at low energies

1. Interactions

Local symmetries

2. QED+QCD

Precision theory for $E \ll 100$ GeV

Qualitative difference QED \iff QCD

3. Chiral symmetry

Some of the quarks happen to be light

Approximate chiral symmetry

Spontaneous symmetry breakdown

4. Goldstone theorem

If N_f of the quark masses are put equal to zero

QCD contains $N_f^2 - 1$ Goldstone bosons

5. Gell-Mann-Oakes-Renner relation

Quark masses break chiral symmetry

Goldstone bosons pick up mass

M_π^2 is proportional to $m_u + m_d$

II. Chiral perturbation theory

6. Group geometry

Symmetry group of the Hamiltonian G

Symmetry group of the ground state H

Goldstone bosons live on G/H

7. Effective action

Generating functional of QCD

8. Ward identities

Symmetries of the effective action of QCD.

9. Low energy expansion

Taylor series in powers of external momenta

Goldstone bosons \Rightarrow infrared singularities

10. Effective Lagrangian

Singularities due to the Goldstone bosons can be worked out with an effective field theory.

Side remark: for nonrelativistic systems, there is a complication. In that case, \mathcal{L}_{eff} is in general invariant only up to a total derivative.

11. Explicit construction of \mathcal{L}_{eff}

III. Illustrations

12. Some tree level calculations

Leading terms of the chiral perturbation series for the quark condensate and for M_π, F_π

13. M_π beyond tree level

Contributions to M_π at NL and NNL orders

14. F_π to one loop

Chiral logarithm in F_π , low energy theorem for scalar radius

15. Lattice results for M_π, F_π

Determination of the effective coupling constants l_3, l_4 on the lattice

16. $\pi\pi$ scattering

χ PT, lattice, experiment

17. Conclusions for $SU(2) \times SU(2)$

18. Expansion in powers of m_s

Convergence, validity of Zweig rule

19. Conclusions for $SU(3) \times SU(3)$

20. Puzzling results on $K_L \rightarrow \pi\mu\nu$

The result of a recent high statistics experiment is in conflict with the Callan-Treiman relation. If confirmed, this indicates physics beyond the Standard Model: right-handed couplings of the W .

Exercises

I. Standard Model at low energies

1. Interactions

strong weak e.m. gravity

$$SU(3) \times SU(2) \times U(1) \times D$$

Gravity

understood only at classical level

gravitational waves ✓

quantum theory of gravity ?

classical theory should be adequate for

$$r \gg \sqrt{\frac{G \hbar}{c^3}} = 1.6 \cdot 10^{-35} \text{ m}$$

Weak interaction

frozen at low energies

$$E \ll M_w c^2 \simeq 80 \text{ GeV}$$

⇒ structure of matter: only strong and electromagnetic interaction

⇒ neutrini decouple

Electromagnetic interaction

Maxwell \sim 1860

survived relativity and quantum theory, unharmed

- Electrons in electromagnetic field ($\hbar = c = 1$)

$$\frac{1}{i} \frac{\partial \psi}{\partial t} - \frac{1}{2m_e^2} (\vec{\nabla} + ie\vec{A})^2 \psi - e\varphi \psi = 0$$

contains the potentials \vec{A} , φ

- only $\vec{E} = -\vec{\nabla}\varphi - \frac{\partial \vec{A}}{\partial t}$ and $\vec{B} = \vec{\nabla} \times \vec{A}$

are of physical significance

- Schrödinger equation is invariant under gauge transformations

$$\vec{A}' = \vec{A} + \vec{\nabla}f, \quad \varphi' = \varphi - \frac{\partial f}{\partial t}, \quad \psi' = e^{-ief} \psi$$

describe the same physical situation as \vec{A}, φ, ψ

- Equivalence principle of the e.m. interaction:

$$\psi \text{ physically equivalent to } e^{-ief} \psi$$

- e^{-ief} is unitary 1×1 matrix, $e^{-ief} \in U(1)$
 $f = f(\vec{x}, t)$ space-time dependent function
- gauge invariance \iff local $U(1)$ symmetry
electromagnetic field is gauge field of $U(1)$
Weyl 1929
- $U(1)$ symmetry + renormalizability
fully determine the e.m. interaction

Strong interaction

nuclei = p + n ~ 1930

- Nuclear forces

Yukawa ~ 1935

$$V_{e.m.} = -\frac{e^2}{4\pi r}$$

$$V_s = -\frac{h^2}{4\pi r} e^{-\frac{r}{r_0}}$$

$$\frac{e^2}{4\pi} \simeq \frac{1}{137}$$

$$\frac{h^2}{4\pi} \simeq 13$$

long range

short range

$$r_0 = \infty$$

$$r_0 = \frac{\hbar}{M_\pi c} = 1.4 \cdot 10^{-15} \text{ m}$$

$$M_\gamma = 0$$

$$M_\pi c^2 \simeq 140 \text{ MeV}$$

- Problem with Yukawa formula:

p and n are extended objects

diameter comparable to range of force

formula only holds for $r \gg$ diameter

- Protons, neutrons composed of quarks

$$p = uud \quad n = udd$$

- Quarks carry internal quantum number

$$u = \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} \quad d = \begin{pmatrix} d_1 \\ d_2 \\ d_3 \end{pmatrix}$$

occur in 3 “colours”

- Strong interaction is invariant under local rotations in colour space 1973

$$u' = U \cdot u \quad d' = U \cdot d$$

$$U = \begin{pmatrix} U_{11} & U_{12} & U_{13} \\ U_{21} & U_{22} & U_{23} \\ U_{31} & U_{32} & U_{33} \end{pmatrix} \in \mathbf{SU}(3)$$

- Can only be so if the strong interaction is also mediated by a gauge field

gauge field of $\mathbf{SU}(3) \implies$ strong interaction

Quantum chromodynamics

Comparison of e.m. and strong interaction

	QED	QCD
symmetry	U(1)	SU(3)
gauge field	\vec{A}, φ	gluon field
particles	photons	gluons
source	charge	colour
coupling constant	e	g

- All charged particles generate e.m. field
- All coloured particles generate gluon field
- Leptons do not interact strongly because they do not carry colour
- Equivalence principle of the strong interaction:

$$U \cdot \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} \text{ physically equivalent to } \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix}$$

2. QED+QCD

Effective theory for $E \ll M_{\text{W}}c^2 \simeq 80 \text{ GeV}$

Symmetry $U(1) \times SU(3)$

Lagrangian QED+QCD

- Dynamical variables:
gauge fields for photons and gluons
Fermi fields for leptons and quarks
- Interaction fully determined by group geometry
Lagrangian contains 2 coupling constants

$$e, g$$

- Quark and lepton mass matrices can be brought to diagonal form, eigenvalues real, positive

$$m_e, m_\mu, m_\tau, m_u, m_d, m_s, m_c, m_b, m_t$$

- Transformation generates vacuum angle

$$\theta$$

- θ breaks CP

Neutron dipole moment is very small

⇒ strong upper limit, $\theta \simeq 0$

⇒ Precision theory for cold matter,
atomic structure, solids, ...

Bohr radius:
$$a = \frac{4\pi}{e^2 m_e}$$

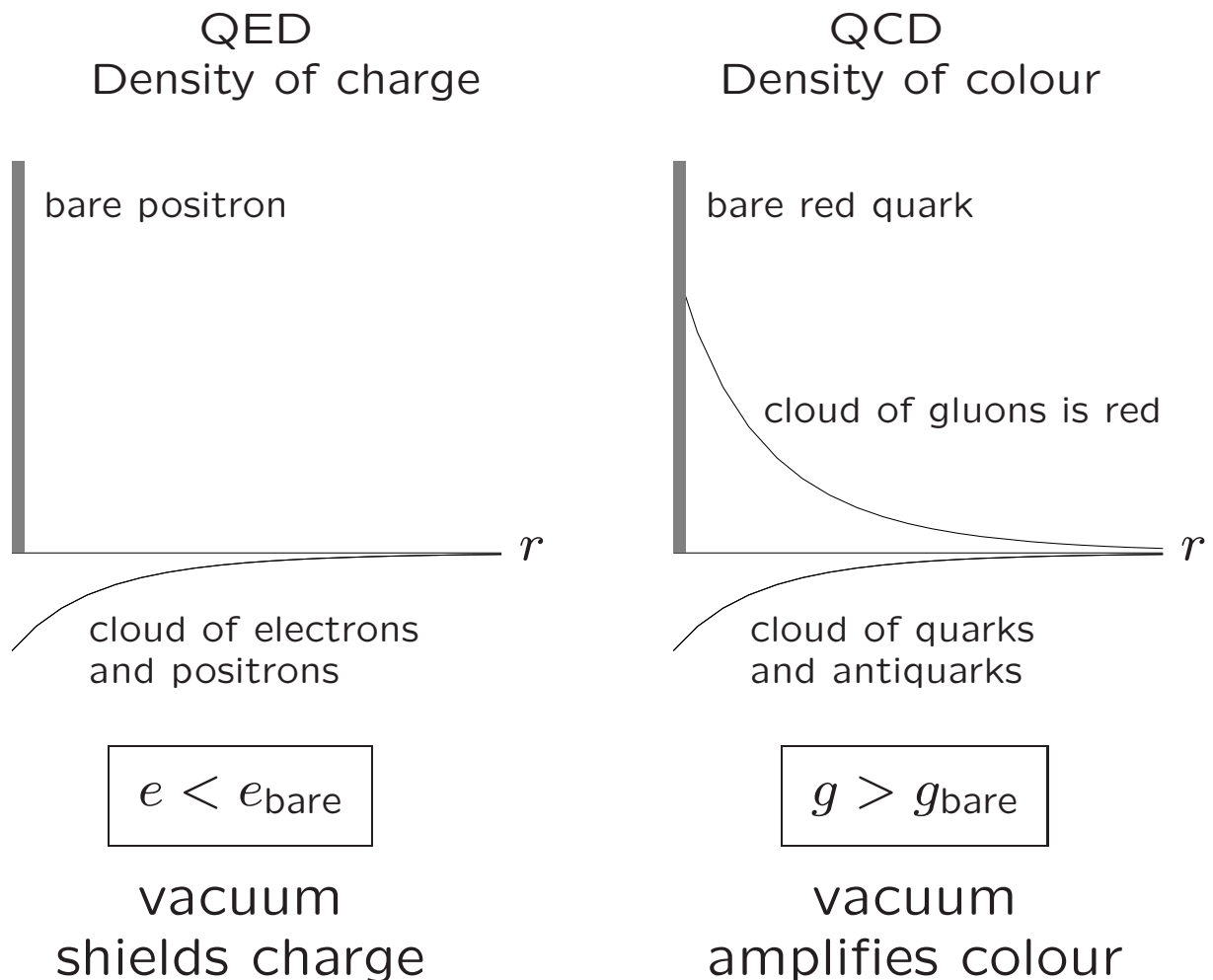
Qualitative difference between e.m. and strong interactions

- Photons do not have charge
- Gluons do have colour

$x_1 \cdot x_2 = x_2 \cdot x_1$ for $x_1, x_2 \in U(1)$ abelian

$x_1 \cdot x_2 \neq x_2 \cdot x_1$ for $x_1, x_2 \in SU(3)$

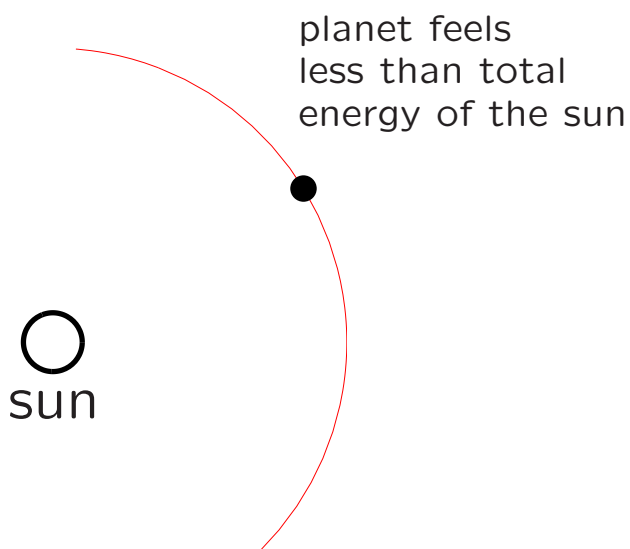
⇒ Consequence for vacuum polarization



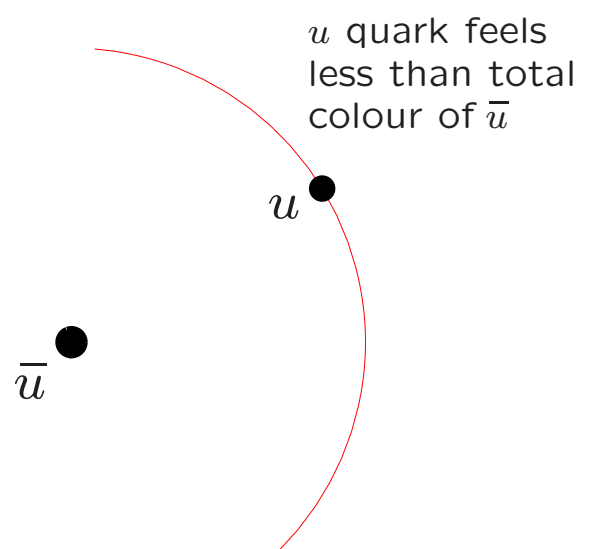
Comparison with gravity

- source of gravitational field: **energy**
gravitational field does carry **energy**
- source of e.m. field: **charge**
e.m. field does not carry **charge**
- source of gluon field: **colour**
gluon field does carry **colour**

gravity



strong interaction



Perihelion shift of Mercury:

$$43'' = 50'' - 7'' \text{ per century}$$

↑

- Force between u and \bar{u} :

$$V_s = -\frac{4}{3} \frac{g^2}{4\pi r}, \quad g \rightarrow 0 \quad \text{for} \quad r \rightarrow 0$$

$$\frac{g^2}{4\pi} = \frac{6\pi}{(11N_c - 2N_f) |\ln(r \Lambda_{\text{QCD}})|}$$

$$|\ln(r \Lambda_{\text{QCD}})| \simeq 7 \quad \text{for} \quad r = \frac{\hbar}{M_Z c} \simeq 2 \cdot 10^{-18} \text{ m}$$

- Vacuum amplifies gluonic field of a bare quark
 - Field energy surrounding isolated quark = ∞
Only colour neutral states have finite energy
- \Rightarrow Confinement of colour
- Theoretical evidence for confinement meagre
Experimental evidence much more convincing

QED: interaction weak at low energies
QCD: interaction strong at low energies

$$\frac{e^2}{4\pi} \simeq \frac{1}{137}$$

photons, leptons
nearly decouple

$$\frac{g^2}{4\pi} \simeq 1$$

gluons, quarks
confined

- Nuclear forces = van der Waals forces of QCD

3. Chiral symmetry

- For bound states of quarks, e.m. interaction is a small perturbation

Perturbation series in powers of $\frac{e^2}{4\pi}$ ✓

Discuss only the leading term: set $e = 0$

- Lagrangian then reduces to QCD

$$g, m_u, m_d, m_s, m_c, m_b, m_t$$

- m_u, m_d, m_s happen to be light

Consequence:

Approximate flavour symmetries

Play a crucial role for the low energy properties

Theoretical paradise

$$m_u = m_d = m_s = 0$$

$$m_c = m_b = m_t = \infty$$

QCD with 3 massless quarks

- Lagrangian contains a single parameter: g
 g is net colour of a quark
depends on radius of the region considered

- Colour contained within radius r

$$\frac{g^2}{4\pi} = \frac{2\pi}{9 |\ln(r \Lambda_{\text{QCD}})|}$$

- Intrinsic scale Λ_{QCD} is meaningful,
but not dimensionless

⇒ No dimensionless free parameter

All dimensionless physical quantities are pure numbers, determined by the theory

Cross sections can be expressed in terms of Λ_{QCD} or in the mass of the proton

- Interactions of u, d, s are identical
If the masses are set equal to zero,
there is no difference at all

$$q = \begin{pmatrix} u \\ d \\ s \end{pmatrix}$$

- Lagrangian symmetric under $u \leftrightarrow d \leftrightarrow s$

$$q' = V \cdot q \quad V \in \text{SU}(3)$$

V acts on quark flavour, mixes u, d, s

- More symmetry: For massless fermions,
right and left do not communicate

⇒ Lagrangian of massless QCD is invariant under independent rotations of the right- and left-handed quark fields

$$q_R = \frac{1}{2}(1 + \gamma_5) q, \quad q_L = \frac{1}{2}(1 - \gamma_5) q$$

$$q'_R = V_R \cdot q_R \quad q'_L = V_L \cdot q_L$$

$$G = \text{SU}(3)_R \times \text{SU}(3)_L$$

- Massless QCD invariant under

$$G = \text{SU}(3)_R \times \text{SU}(3)_L$$

$\text{SU}(3)$ has 8 parameters

⇒ Symmetry under Lie group with 16 parameters

⇒ 16 conserved “charges”

Q_1^V, \dots, Q_8^V (vector currents)

Q_1^A, \dots, Q_8^A (axial currents)

commute with the Hamiltonian:

$$[Q_i^V, H_0] = 0 \quad [Q_i^A, H_0] = 0$$

“Chiral symmetry” of massless QCD

- Vafa and Witten 1984: state of lowest energy is invariant under the vector charges

$$Q_i^V |0\rangle = 0$$

- Axial charges ? $Q_i^A |0\rangle = ?$

Two alternatives for axial charges

$$Q_i^A |0\rangle = 0$$

Wigner-Weyl realization of G
ground state is symmetric

$$\langle 0 | \bar{q}_R q_L | 0 \rangle = 0$$

ordinary symmetry
spectrum contains parity partners
degenerate multiplets of G

$$Q_i^A |0\rangle \neq 0$$

Nambu-Goldstone realization of G
ground state is asymmetric

$$\langle 0 | \bar{q}_R q_L | 0 \rangle \neq 0$$

“order parameter”
spontaneously broken symmetry
spectrum contains Goldstone bosons
degenerate multiplets of $SU(3)_V \subset G$

$$G = SU(3)_R \times SU(3)_L$$

- Spontaneous symmetry breakdown was discovered in condensed matter physics:
Spontaneous magnetization selects direction
- ⇒ Rotation symmetry is spontaneously broken
Goldstone bosons = spin waves, magnons
- Nambu 1960: state of lowest energy in particle physics is not invariant under chiral rotations
 $Q_i^A |0\rangle \neq 0$
For dynamical reasons, the state of lowest energy must be asymmetric
- ⇒ Chiral symmetry is spontaneously broken
- Very strong experimental evidence ✓
- Theoretical understanding on the basis of the QCD Lagrangian ?

- Analog of Magnetization ?

$$\bar{q}_R q_L = \begin{pmatrix} \bar{u}_R u_L & \bar{d}_R u_L & \bar{s}_R u_L \\ \bar{u}_R d_L & \bar{d}_R d_L & \bar{s}_R d_L \\ \bar{u}_R s_L & \bar{d}_R s_L & \bar{s}_R s_L \end{pmatrix}$$

Transforms like $(\bar{3}, 3)$ under $SU(3)_R \times SU(3)_L$

If the ground state were symmetric, the matrix $\langle 0 | \bar{q}_R q_L | 0 \rangle$ would have to vanish, because it singles out a direction in flavour space

“quark condensate”, is quantitative measure of spontaneous symmetry breaking

“order parameter”

$$\langle 0 | \bar{q}_R q_L | 0 \rangle \Leftrightarrow \text{magnetization}$$

- Ground state is invariant under $SU(3)_V$

$\Rightarrow \langle 0 | \bar{q}_R q_L | 0 \rangle$ is proportional to unit matrix

$$\langle 0 | \bar{u}_R u_L | 0 \rangle = \langle 0 | \bar{d}_R d_L | 0 \rangle = \langle 0 | \bar{s}_R s_L | 0 \rangle$$

$$\langle 0 | \bar{u}_R d_L | 0 \rangle = \dots = 0$$

4. Goldstone Theorem

- Consequence of $Q_i^A |0\rangle \neq 0$:

$$H_0 Q_i^A |0\rangle = Q_i^A H_0 |0\rangle = 0$$

spectrum must contain 8 states

$$Q_1^A |0\rangle, \dots, Q_8^A |0\rangle \quad \text{with } E = 0,$$

spin 0, negative parity, octet of $SU(3)_V$

Goldstone bosons

- Argument is not water tight:

$$\langle 0 | Q_i^A Q_k^A | 0 \rangle = \int d^3x d^3y \langle 0 | A_i^0(x) A_k^0(y) | 0 \rangle$$

$$\langle 0 | A_i^0(x) A_k^0(y) | 0 \rangle \text{ only depends on } \vec{x} - \vec{y}$$

$\Rightarrow \langle 0 | Q_i^A Q_k^A | 0 \rangle$ is proportional to the volume of the universe, $|\langle Q_i^A | 0 \rangle| = \infty$

- Rigorous version of Goldstone theorem:
 $\langle 0 | \bar{q}_R q_L | 0 \rangle \neq 0 \Rightarrow \exists$ massless particles

Proof

$$Q = \int d^3x \bar{u} \gamma^0 \gamma_5 d$$

$$[Q, \bar{d} \gamma_5 u] = -\bar{u}u - \bar{d}d$$

- $F^\mu(x - y) \equiv \langle 0 | \bar{u}(x) \gamma^\mu \gamma_5 d(x) \bar{d}(y) \gamma_5 u(y) | 0 \rangle$
 Lorentz invariance $\Rightarrow F^\mu(z) = z^\mu f(z^2)$
 Chiral symmetry $\Rightarrow \partial_\mu F^\mu(z) = 0$

$$F^\mu(z) = \frac{z^\mu}{z^4} \times \text{constant (for } z^2 \neq 0)$$

- Spectral decomposition:

$$F^\mu(x - y) = \langle 0 | \bar{u}(x) \gamma^\mu \gamma_5 d(x) \bar{d}(y) \gamma_5 u(y) | 0 \rangle$$

$$= \sum_n \langle 0 | \bar{u} \gamma^\mu \gamma_5 d | n \rangle \langle n | \bar{d} \gamma_5 u | 0 \rangle e^{-i p_n(x-y)}$$

$p_n^0 \geq 0 \Rightarrow F^\mu(z)$ is analytic in z^0 for $\text{Im } z^0 < 0$

$$F^\mu(z) = \frac{z^\mu}{\{(z^0 - i\epsilon)^2 - \vec{z}^2\}^2} \times \text{constant}$$

- Positive frequency part of massless propagator:

$$\Delta^+(z, 0) = \frac{i}{(2\pi)^3} \int \frac{d^3p}{2p^0} e^{-ipz} \quad , \quad p^0 = |\vec{p}|$$

$$= \frac{1}{4\pi i \{(z^0 - i\epsilon)^2 - \vec{z}^2\}}$$

- Result

$$\langle 0 | \bar{u}(x) \gamma^\mu \gamma_5 d(x) \bar{d}(y) \gamma_5 u(y) | 0 \rangle = C \partial^\mu \Delta^+(z, 0)$$

- Compare Källen–Lehmann representation:

$$\langle 0 | \bar{u}(x) \gamma^\mu \gamma_5 d(x) \bar{d}(y) \gamma_5 u(y) | 0 \rangle$$

$$= (2\pi)^{-3} \int d^4p p^\mu \rho(p^2) e^{-ip(x-y)}$$

$$= \int_0^\infty ds \rho(s) \partial^\mu \Delta^+(x-y, s)$$

$\Delta^+(z, s) \iff$ massive propagator

$$\Delta^+(z, s) = \frac{i}{(2\pi)^3} \int d^4p \theta(p^0) \delta(p^2 - s) e^{-ipz}$$

\Rightarrow Only massless intermediate states contribute:

$$\rho(s) = C \delta(s)$$

- Why only massless intermediate states ?

$\langle n | \bar{d} \gamma_5 u | 0 \rangle \neq 0$ only if $\langle n |$ has spin 0

If $|n\rangle$ has spin 0 $\Rightarrow \langle 0 | \bar{u}(x) \gamma^\mu \gamma_5 d(x) | n \rangle \propto p^\mu e^{-ipx}$

$\partial_\mu (\bar{u} \gamma^\mu \gamma_5 d) = 0 \Rightarrow p^2 = 0$

\Rightarrow Either \exists massless particles or $C = 0$

- Claim: $\langle 0 | \bar{q}_R q_L | 0 \rangle \neq 0 \Rightarrow C \neq 0$

Lorentz invariance, chiral symmetry

$\Rightarrow \langle 0 | \bar{d}(y) \gamma_5 u(y) \bar{u}(x) \gamma^\mu \gamma_5 d(x) | 0 \rangle = C' \partial^\mu \Delta^-(z)$

$\Rightarrow \langle 0 | [\bar{u}(x) \gamma^\mu \gamma_5 d(x), \bar{d}(y) \gamma_5 u(y)] | 0 \rangle$

$$= C \partial^\mu \Delta^+(z, 0) - C' \partial^\mu \Delta^-(z, 0)$$

- Causality: if $x - y$ is spacelike, then

$\langle 0 | [\bar{u}(x) \gamma^\mu \gamma_5 d(x), \bar{d}(y) \gamma_5 u(y)] | 0 \rangle = 0$

$\Rightarrow C' = -C$

$\Rightarrow \langle 0 | [\bar{u}(x) \gamma^\mu \gamma_5 d(x), \bar{d}(y) \gamma_5 u(y)] | 0 \rangle = C \partial^\mu \Delta(z, 0)$

$\Rightarrow \langle 0 | [Q, \bar{d}(y) \gamma_5 u(y)] | 0 \rangle = C$

- $\langle 0 | [Q, \bar{d}(y) \gamma_5 u(y)] | 0 \rangle = -\langle 0 | \bar{u}u + \bar{d}d | 0 \rangle = C$

Hence $\langle 0 | \bar{u}u + \bar{d}d | 0 \rangle \neq 0$ implies $C \neq 0$ qed.

5. Gell-Mann-Oakes-Renner relation

⇒ Spectrum of QCD with 3 massless quarks must contain 8 massless physical particles, $J^P = 0^-$

- Indeed, the 8 lightest mesons do have these quantum numbers:

$$\pi^+, \pi^0, \pi^-, K^+, K^0, \bar{K}^0, K^-, \eta$$

But massless they are not

- Real world \neq paradise

$$m_u, m_d, m_s \neq 0$$

Quark masses break chiral symmetry,
allow left to talk to right

- Chiral symmetry broken in two ways:

spontaneously $\langle 0 | \bar{q}_R q_L | 0 \rangle \neq 0$

explicitly $m_u, m_d, m_s \neq 0$

- H_{QCD} only has approximate symmetry to the extent that m_u, m_d, m_s are small

$$H_{\text{QCD}} = H_0 + H_1$$

$$H_1 = \int d^3x \{m_u \bar{u}u + m_d \bar{d}d + m_s \bar{s}s\}$$

- H_0 is Hamiltonian of the massless theory, invariant under $SU(3)_R \times SU(3)_L$
- H_1 breaks the symmetry, transforms with $(3, \bar{3}) \oplus (\bar{3}, 3)$
- For the low energy structure of QCD, the heavy quarks do not play an essential role: c, b, t are singlets under $SU(3)_R \times SU(3)_L$
Can include the heavy quarks in H_0
- Goldstone bosons are massless only if the symmetry is exact

$$M_{\pi}^2 = (m_u + m_d) \times |\langle 0 | \bar{u} u | 0 \rangle| \times \frac{1}{F_{\pi}^2}$$

1968

↑

↑

explicit spontaneous

Coefficient: decay constant F_{π}

Derivation

- Pion matrix elements in massless theory:

$$\begin{aligned} \langle 0 | \bar{u} \gamma^{\mu} \gamma_5 d | \pi^{-} \rangle &= i \sqrt{2} F p^{\mu} \\ \langle 0 | \bar{u} i \gamma_5 d | \pi^{-} \rangle &= \sqrt{2} G \end{aligned}$$

Only the one-pion intermediate state

$$\langle 0 | \bar{u}(x) \gamma^{\mu} \gamma_5 d(x) \overset{\uparrow}{| \pi^{-} \rangle \langle \pi^{-} |} \bar{d}(y) \gamma_5 u(y) | 0 \rangle = C \partial^{\mu} \Delta^{+}(z, 0)$$

contributes. Hence $2 F G = C$

- Value of C fixed by quark condensate

$$C = -\langle 0 | \bar{u} u + \bar{d} d | 0 \rangle$$

⇒ Exact result in massless theory:

$$F G = -\langle 0 | \bar{u} u | 0 \rangle$$

- Matrix elements for $m_{\text{quark}} \neq 0$:

$$\langle 0 | \bar{u} \gamma^\mu \gamma_5 d | \pi^- \rangle = i \sqrt{2} F_\pi p^\mu$$

$$\langle 0 | \bar{u} i \gamma_5 d | \pi^- \rangle = \sqrt{2} G_\pi$$

- Current conservation

$$\partial_\mu (\bar{u} \gamma^\mu \gamma_5 d) = (m_u + m_d) \bar{u} i \gamma_5 d$$

$$\Rightarrow F_\pi M_\pi^2 = (m_u + m_d) G_\pi$$

$$\boxed{M_\pi^2 = (m_u + m_d) \frac{G_\pi}{F_\pi}} \quad \text{exact for } m \neq 0$$

- $F_\pi \rightarrow F$, $G_\pi \rightarrow G$ for $m \rightarrow 0$

$$F G = -\langle 0 | \bar{u} u | 0 \rangle$$

$$\Rightarrow \frac{G_\pi}{F_\pi} = -\frac{\langle 0 | \bar{u} u | 0 \rangle}{F_\pi^2} + O(m)$$

$$\Rightarrow M_\pi^2 = (m_u + m_d) \left(\frac{-\langle 0 | \bar{u} u | 0 \rangle}{F_\pi^2} \right) + O(m^2) \quad \checkmark$$

$$\Rightarrow \langle 0 | \bar{u} u | 0 \rangle \leq 0 \text{ if quark masses are positive}$$

- $M_\pi^2 = (m_u + m_d) B + O(m^2)$

$$B = \frac{|\langle 0 | \bar{u} u | 0 \rangle|}{F_\pi^2}$$

- M_π disappears if the symmetry breaking is turned off, $m_u, m_d \rightarrow 0$ ✓

- Explains why the pseudoscalar mesons have very different masses

$$M_{K^+}^2 = (m_u + m_s) B + O(m^2)$$

$$M_{K^-}^2 = (m_d + m_s) B + O(m^2)$$

⇒ M_K^2 is about 13 times larger than M_π^2 , because m_u, m_d happen to be small compared to m_s

- First order perturbation theory also yields

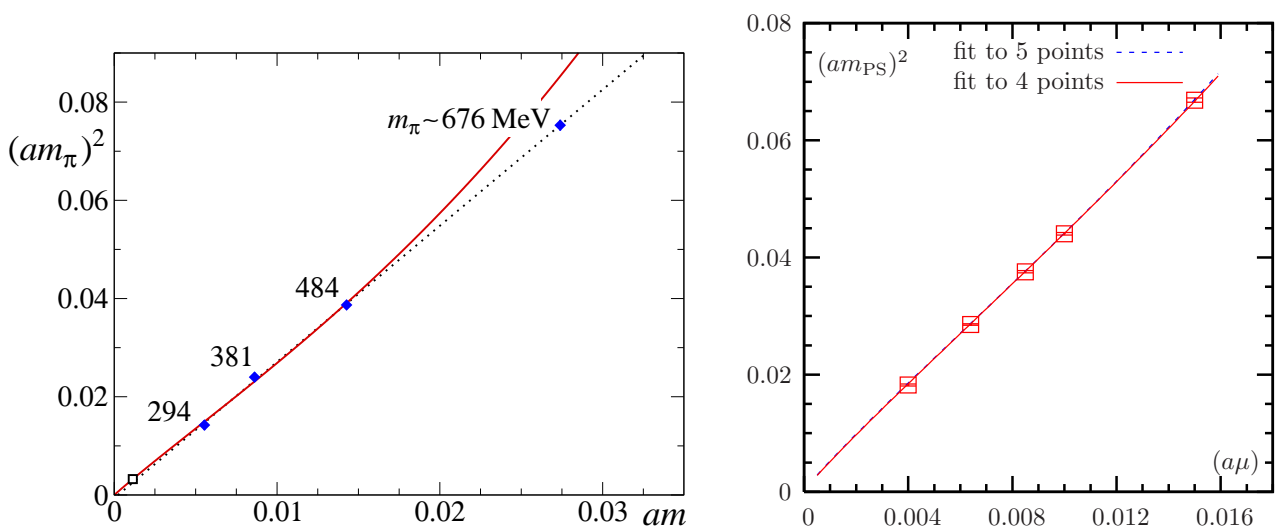
$$M_\eta^2 = \frac{1}{3} (m_u + m_d + 4m_s) B + O(m^2)$$

⇒ $M_\pi^2 - 4M_K^2 + 3M_\eta^2 = O(m^2)$

Gell-Mann-Okubo formula for M^2 ✓

Checking the GMOR formula on a lattice

- Can determine M_π as function of $m_u = m_d = m$



Lüscher, Lattice conference 2005 ETM collaboration, hep-lat/0701012

- No quenching, quark masses sufficiently light
- ⇒ Legitimate to use χ Pt for the extrapolation to the physical values of m_u, m_d

- Quality of data is impressive
- Proportionality of M_π^2 to the quark mass appears to hold out to values of m_u, m_d that are an order of magnitude larger than in nature
- Main limitation: systematic uncertainties
in particular: $N_f = 2 \rightarrow N_f = 3$

II. Chiral perturbation theory

6. Group geometry

- QCD with 3 massless quarks:
spontaneous symmetry breakdown
from $SU(3)_R \times SU(3)_L$ to $SU(3)_V$
generates 8 Goldstone bosons
- Generalization: suppose symmetry group
of Hamiltonian is Lie group G
Generators $Q_1, Q_2, \dots, Q_D, D = \dim(G)$
For some generators $Q_i |0\rangle \neq 0$
How many Goldstone bosons ?
- Consider those elements of the Lie algebra
 $Q = \alpha_1 Q_1 + \dots + \alpha_n Q_D$, for which $Q |0\rangle = 0$
These elements form a subalgebra:
 $Q |0\rangle = 0, Q' |0\rangle = 0 \Rightarrow [Q, Q'] |0\rangle = 0$
Dimension of subalgebra: $d \leq D$
- Of the D vectors $Q_i |0\rangle$
 $D - d$ are linearly independent
 $\Rightarrow D - d$ different physical states of zero mass
 $\Rightarrow D - d$ Goldstone bosons

- Subalgebra generates subgroup $H \subset G$
 H is symmetry group of the ground state
 coset space G/H contains as many parameters
 as there are Goldstone bosons
 $d = \dim(H)$, $D = \dim(G)$

⇒ Goldstone bosons live on the coset G/H

- Example: QCD with N_f massless quarks

$$G = SU(N_f)_R \times SU(N_f)_L$$

$$H = SU(N_f)_V$$

$$D = 2(N_f^2 - 1), \quad d = N_f^2 - 1$$

$$N_f^2 - 1 \text{ Goldstone bosons}$$

- It so happens that $m_u, m_d \ll m_s$
- $m_u = m_d = 0$ is an excellent approximation
 $SU(2)_R \times SU(2)_L$ is a nearly exact symmetry
 $N_f = 2$, $N_f^2 - 1 = 3$ Goldstone bosons (pions)

7. Effective action

- Basic objects for quantitative analysis of QCD: Green functions of the currents

$$V_a^\mu = \bar{q} \gamma^\mu \frac{1}{2} \lambda_a q, \quad A_a^\mu = \bar{q} \gamma^\mu \gamma_5 \frac{1}{2} \lambda_a q,$$

$$S_a = \bar{q} \frac{1}{2} \lambda_a q, \quad P_a = \bar{q} i \gamma_5 \frac{1}{2} \lambda_a q$$

Include singlets, with $\lambda_0 = \sqrt{2/3} \times \mathbf{1}$, as well as

$$\omega = \frac{1}{16\pi^2} \text{tr}_c G_{\mu\nu} \tilde{G}^{\mu\nu}$$

- Can collect all of the Green functions formed with these operators in a generating functional: Perturb the system with external fields $v_\mu^a(x), a_\mu^a(x), s_a(x), p^a(x), \theta(x)$

Replace the Lagrangian of the massless theory

$$\mathcal{L}_0 = -\frac{1}{2g^2} \text{tr}_c G_{\mu\nu} G^{\mu\nu} + \bar{q} i \gamma^\mu (\partial_\mu - i G_\mu) q$$

by $\mathcal{L} = \mathcal{L}_0 + \mathcal{L}_1$

$$\mathcal{L}_1 = v_\mu^a V_a^\mu + a_\mu^a A_a^\mu - s^a S_a - p^a P_a - \theta \omega$$

- Quark mass terms are included in the external field $s_a(x)$

- $|0 \text{ in}\rangle$: system is in ground state for $x^0 \rightarrow -\infty$
Probability amplitude for finding ground state when $x^0 \rightarrow +\infty$:

$$e^{iS_{\text{QCD}}\{v,a,s,p,\theta\}} = \langle 0 \text{ out} | 0 \text{ in}\rangle_{v,a,s,p,\theta}$$

- Expressed in terms of ground state of \mathcal{L}_0 :

$$e^{iS_{\text{QCD}}\{v,a,s,p,\theta\}} = \langle 0 | T \exp i \int dx \mathcal{L}_1 | 0 \rangle$$

- Expansion of $S_{\text{QCD}}\{v, a, s, p, \theta\}$ in powers of the external fields yields the connected parts of the Green functions of the massless theory

$$S_{\text{QCD}}\{v, a, s, p, \theta\} = - \int dx s_a(x) \langle 0 | S^a(x) | 0 \rangle + \frac{i}{2} \int dx dy a_\mu^a(x) a_\nu^b(y) \langle 0 | T A_a^\mu(x) A_b^\nu(y) | 0 \rangle_{\text{conn}} + \dots$$

- For Green functions of full QCD, set

$$s_a(x) = m_a + \tilde{s}_a(x), \quad m_a = \text{tr} \lambda_a m$$

and expand around $\tilde{s}_a(x) = 0$

- Path integral representation of effective action:

$$e^{iS_{\text{QCD}}\{v,a,s,p\}} = \mathcal{N} \int [dG] e^{i \int dx \mathcal{L}_G} \det D$$

$$\mathcal{L}_G = -\frac{1}{2g^2} \text{tr}_c G_{\mu\nu} G^{\mu\nu} - \frac{\theta}{16\pi^2} \text{tr}_c G_{\mu\nu} \tilde{G}^{\mu\nu}$$

$$D = i\gamma^\mu \{ \partial_\mu - i(G_\mu + v_\mu + a_\mu \gamma_5) \} - s - i\gamma_5 p$$

G_μ is matrix in colour space

v_μ, a_μ, s, p are matrices in flavour space

$$v_\mu(x) \equiv \frac{1}{2} \lambda_a v_\mu^a(x), \text{ etc.}$$

8. Ward identities

Symmetry in terms of Green functions

- Lagrangian is invariant under

$$q_R(x) \rightarrow V_R(x) q_R(x), \quad q_L(x) \rightarrow V_L(x) q_L(x)$$
$$V_R(x), V_L(x) \in U(3)$$

provided the external fields are transformed with

$$v'_\mu + a'_\mu = V_R(v_\mu + a_\mu)V_R^\dagger - i\partial_\mu V_R V_R^\dagger$$
$$v'_\mu - a'_\mu = V_L(v_\mu - a_\mu)V_L^\dagger - i\partial_\mu V_L V_L^\dagger$$
$$s' + i p' = V_R(s + i p)V_L^\dagger$$

The operation takes the Dirac operator into

$$D' = \{P_- V_R + P_+ V_L\} D \{P_+ V_R^\dagger + P_- V_L^\dagger\}$$
$$P_\pm = \frac{1}{2}(1 \pm \gamma_5)$$

- $\det D$ requires regularization

⚡ symmetric regularization

$$\Rightarrow \det D' \neq \det D, \text{ only } |\det D'| = |\det D|$$

symmetry does not survive quantization

- Change in $\det D$ can explicitly be calculated

For an infinitesimal transformation

$$V_R = 1 + i\alpha + i\beta + \dots, \quad V_L = 1 + i\alpha - i\beta + \dots$$

the change in the determinant is given by

$$\det D' = \det D e^{-i \int dx \{2\langle\beta\rangle\omega + \langle\beta\Omega\rangle\}}$$

$$\langle A \rangle \equiv \text{tr } A$$

$$\omega = \frac{1}{16\pi^2} \text{tr}_c G_{\mu\nu} \tilde{G}^{\mu\nu} \quad \text{gluons}$$

$$\Omega = \frac{N_c}{4\pi^2} \epsilon^{\mu\nu\rho\sigma} \partial_\mu v_\nu \partial_\rho v_\sigma + \dots \quad \text{ext. fields}$$

- Consequence for effective action:

The term with ω amounts to a change in θ ,
can be compensated by $\theta' = \theta - 2\langle\beta\rangle$

Pull term with $\langle\beta\Omega\rangle$ outside the path integral

$$\Rightarrow S_{\text{QCD}}\{v', a', s', p', \theta'\} = S_{\text{QCD}}\{v, a, s, p, \theta\} - \int dx \langle\beta\Omega\rangle$$

$$S_{\text{QCD}}\{v', a', s', p', \theta'\} = S_{\text{QCD}}\{v, a, s, p, \theta\} - \int dx \langle \beta \Omega \rangle$$

- S_{QCD} is invariant under $U(3)_R \times U(3)_L$, except for a specific change due to the anomalies
- Relation plays key role in low energy analysis: collects all of the Ward identities
For the octet part of the axial current, e.g.

$$\begin{aligned} \partial_\mu^x \langle 0 | T A_a^\mu(x) P_b(y) | 0 \rangle &= -\frac{1}{4} i \delta(x - y) \langle 0 | \bar{q} \{ \lambda_a, \lambda_b \} q | 0 \rangle \\ &\quad + \langle 0 | T \bar{q}(x) i \gamma_5 \{ m, \frac{1}{2} \lambda_a \} q(x) P_b(y) | 0 \rangle \end{aligned}$$

- Symmetry of the effective action implies the operator relations

$$\partial_\mu V_a^\mu = \bar{q} i [m, \frac{1}{2} \lambda_a] q, \quad a = 0, \dots, 8$$

$$\partial_\mu A_a^\mu = \bar{q} i \gamma_5 \{ m, \frac{1}{2} \lambda_a \} q, \quad a = 1, \dots, 8$$

$$\partial_\mu A_0^\mu = \sqrt{\frac{2}{3}} \bar{q} i \gamma_5 m q + \sqrt{6} \omega$$

- Textbook derivation of the Ward identities goes in inverse direction, but is slippery formal manipulations, anomalies ?

9. Low energy expansion

- If the spectrum has an energy gap
- ⇒ no singularities in scattering amplitudes or Green functions near $p = 0$
- ⇒ low energy behaviour may be analyzed with Taylor series expansion in powers of p

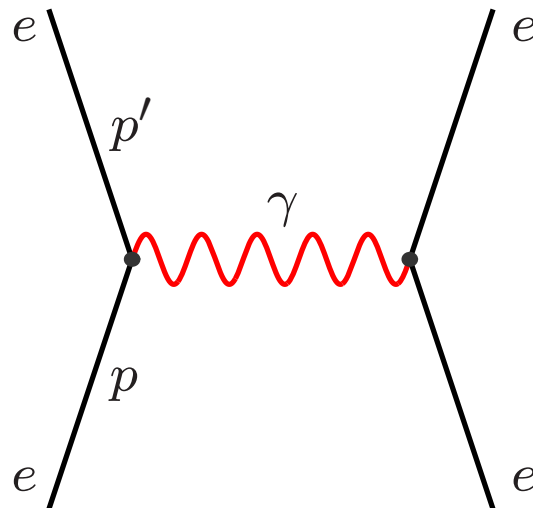
$$f(t) = 1 + \frac{1}{6} \langle r^2 \rangle t + \dots \text{ form factor}$$

$$T(p) = a + b p^2 + \dots \text{ scattering amplitude}$$

Cross section dominated by S -wave scattering length $\frac{d\sigma}{d\Omega} \simeq |a|^2$

- Expansion parameter: $\frac{p}{m} = \frac{\text{momentum}}{\text{energy gap}}$
- Taylor series only works if the spectrum has an energy gap, i.e. if there are no massless particles

- Illustration: Coulomb scattering



Photon exchange \Rightarrow pole at $t = 0$

$$T = \frac{e^2}{(p' - p)^2}$$

Scattering amplitude does not admit Taylor series expansion in powers of p

- QCD does have an energy gap but the gap is very small: M_π
- \Rightarrow Taylor series has very small radius of convergence, useful only for $p < M_\pi$

- Massless QCD contains infrared singularities due to the Goldstone bosons
 - For $m_u = m_d = 0$, pion exchange gives rise to poles and branch points at $p = 0$
- ⇒ Low energy expansion is not a Taylor series, contains logarithms

Singularities due to Goldstone bosons can be worked out with an effective field theory
 “Chiral Perturbation Theory”

Weinberg, Dashen, Pagels, Gasser, . . .

- Chiral perturbation theory correctly reproduces the infrared singularities of QCD
- Quantities of interest are expanded in powers of external momenta and quark masses
- Expansion has been worked out to next-to-leading order for many quantities
 ” Chiral perturbation theory to one loop”
- In quite a few cases, the next-to-next-to-leading order is also known

- Properties of the Goldstone bosons are governed by the hidden symmetry that is responsible for their occurrence
- Focus on the singularities due to the pions
Include the mass term of strange quark in H_0

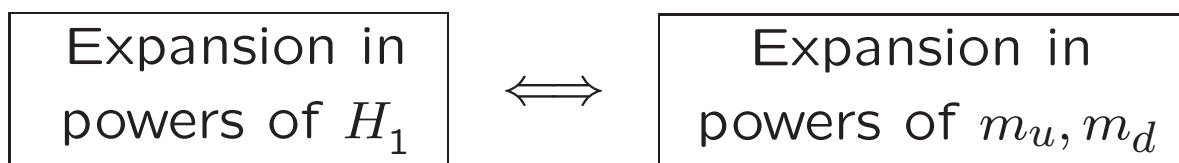
$$H_{\text{QCD}} = H_0 + H_1$$

$$H_1 = \int d^3x \{m_u \bar{u}u + m_d \bar{d}d\}$$

H_0 is invariant under $G = \text{SU}(2)_R \times \text{SU}(2)_L$

$|0\rangle$ is invariant under $H = \text{SU}(2)_V$

- Treat H_1 as a perturbation



- Extension to $\text{SU}(3)_R \times \text{SU}(3)_L$ straightforward:
include singularities due to exchange of K, η

10. Effective Lagrangian

- Replace quarks and gluons by pions

$$\vec{\pi}(x) = \{\pi^1(x), \pi^2(x), \pi^3(x)\}$$

$$\mathcal{L}_{\text{QCD}} \rightarrow \mathcal{L}_{\text{eff}}$$

- Central claim:
 - A. Effective theory yields alternative representation for effective action of QCD

$$e^{iS_{\text{QCD}}\{v,a,s,p,\theta\}} = \mathcal{N}_{\text{eff}} \int [d\pi] e^{i \int dx \mathcal{L}_{\text{eff}}\{\vec{\pi},v,a,s,p,\theta\}}$$

B. \mathcal{L}_{eff} has the same symmetries as \mathcal{L}_{QCD}

⇒ Can calculate the low energy expansion of the Green functions with the effective theory.

If \mathcal{L}_{eff} is chosen properly, this reproduces the low energy expansion of QCD, order by order.

- Proof of A and B: H.L., Annals Phys. 1994

- Pions live on the coset $G/H = SU(2)$

$$\vec{\pi}(x) \rightarrow U(x) \in SU(2)$$

The fields $\vec{\pi}(x)$ are the coordinates of $U(x)$

Can use canonical coordinates, for instance

$$U = \exp i \vec{\pi} \cdot \vec{\tau} \in SU(2)$$

- Action of the symmetry group on the quarks:

$$q'_R = V_R \cdot q_R, \quad q'_L = V_L \cdot q_L$$

- Action on the pion field:

$$U' = V_R \cdot U \cdot V_L^\dagger$$

Note: Transformation law for the coordinates $\vec{\pi}$ is complicated, nonlinear

- Except for the contribution from the anomalies, \mathcal{L}_{eff} is invariant

$$\boxed{\mathcal{L}_{eff}\{U', v', a', s', p', \theta'\} = \mathcal{L}_{eff}\{U, v, a, s, p, \theta\}}$$

Symmetry of S_{QCD} implies symmetry of \mathcal{L}_{eff}

Side remark

- For nonrelativistic effective theories, the effective Lagrangian is in general invariant only up to a total derivative.
- ⇒ From the point of view of effective field theory, nonrelativistic systems with Goldstone bosons are more complicated than relativistic ones

detailed discussion: H. L., Phys. Rev. D49 (1994) 3033

- Origin of the complication: the generators of the symmetry group may themselves give rise to order parameters

$$\langle 0 | Q^i | 0 \rangle \neq 0$$

This cannot happen in the relativistic case:

$$Q = \int d^3x j^0(x)$$
$$\langle 0 | j^\mu(x) | 0 \rangle = 0 \Rightarrow \langle 0 | Q | 0 \rangle = 0$$

Nonrelativistic example where it does happen:

Heisenberg model of a ferromagnet

$$H = -g \sum_{\langle ij \rangle} \vec{s}_i \cdot \vec{s}_j$$

$g > 0$ $\uparrow\uparrow$ lower in energy than $\uparrow\downarrow$

- Ground state = $\uparrow\uparrow\uparrow\uparrow \cdots \uparrow\uparrow$
- Magnetization: $\vec{M} = \frac{\mu}{V} \sum_i \vec{s}_i$
 $\langle 0 | \vec{M} | 0 \rangle \neq 0 \iff \langle 0 | \bar{q}_R q_L | 0 \rangle \neq 0$
- Symmetry generators: $\vec{Q} = \sum_i \vec{s}_i \propto \vec{M}$
- Hamiltonian is invariant under the full rotation group $G = \text{SO}(3)$, ground state is invariant only under rotations around the direction of $\langle 0 | \vec{M} | 0 \rangle$, $H = U(1)$
- Effective field lives on $G/H = S_2$: unit vector \vec{U} , parametrized by 2 coordinates π^1, π^2 .
- Effective Lagrangian of ferromagnet is invariant under local rotations only up to a total derivative. Leading term is related to the Brouwer degree of the map $(\pi^1, \pi^2) \rightarrow \vec{U}$.

11. Explicit construction of \mathcal{L}_{eff}

- First ignore the external fields,

$$\mathcal{L}_{eff} = \mathcal{L}_{eff}(U, \partial U, \partial^2 U, \dots)$$

Derivative expansion:

$$\mathcal{L}_{eff} = f_0(U) + f_1(U) \times \square U + f_2(U) \times \partial_\mu U \times \partial^\mu U + \dots$$

$$\begin{array}{ccc} \uparrow & \uparrow & \uparrow \\ O(1) & O(p^2) & O(p^2) \end{array}$$

Amounts to expansion in powers of momenta

- Term of $O(1)$: $f_0(U) = f_0(V_R U V_L^\dagger)$
 $V_R = \mathbf{1}, \quad V_L = U \rightarrow V_R U V_L^\dagger = \mathbf{1}$
- $\Rightarrow f_0(U) = f_0(\mathbf{1})$ irrelevant constant, drop it
- Term with $\square U$: integrate by parts
- \Rightarrow can absorb $f_1(U)$ in $f_2(U)$

⇒ Derivative expansion of \mathcal{L}_{eff} starts with

$$\mathcal{L}_{eff} = f_2(U) \times \partial_\mu U \times \partial^\mu U + O(p^4)$$

- Replace the partial derivative by

$$\Delta_\mu \equiv \partial_\mu U U^\dagger, \quad \text{tr} \Delta_\mu = 0$$

Δ_μ is invariant under $SU(2)_L$ and transforms with the representation $D^{(1)}$ under $SU(2)_R$:

$$\Delta_\mu \rightarrow V_R \Delta_\mu V_R^\dagger$$

In this notation, leading term is of the form

$$\mathcal{L}_{eff} = \tilde{f}_2(U) \times \Delta_\mu \times \Delta^\mu + O(p^4)$$

- Invariance under $SU(2)_L$: $\tilde{f}_2(U) = \tilde{f}_2(U V_L^\dagger)$
- ⇒ $\tilde{f}_2(U)$ is independent of U
- Invariance under $SU(2)_R$: $\Delta_\mu \times \Delta^\mu$ transforms with $D^{(1)} \times D^{(1)} \rightarrow$ contains unity exactly once: $\text{tr}(\Delta_\mu \Delta^\mu) = \text{tr}(\partial_\mu U U^\dagger \partial^\mu U U^\dagger) = -\text{tr}(\partial_\mu U \partial^\mu U^\dagger)$
- ⇒ Geometry fixes leading term up to a constant

$$\mathcal{L}_{eff} = \frac{F^2}{4} \text{tr}(\partial_\mu U \partial^\mu U^\dagger) + O(p^4)$$

$$\mathcal{L}_{eff} = \frac{F^2}{4} \text{tr}(\partial_\mu U \partial^\mu U^\dagger) + O(p^4)$$

- Lagrangian of the nonlinear σ -model
- Expansion in powers of $\vec{\pi}$:

$$U = \exp i \vec{\pi} \cdot \vec{\tau} = \mathbf{1} + i \vec{\pi} \cdot \vec{\tau} - \frac{1}{2} \vec{\pi}^2 + \dots$$

$$\Rightarrow \mathcal{L}_{eff} = \frac{F^2}{2} \partial_\mu \vec{\pi} \cdot \partial^\mu \vec{\pi} + \frac{F^2}{48} \text{tr}\{[\partial_\mu \pi, \pi] [\partial^\mu \pi, \pi]\} + \dots$$

For the kinetic term to have the standard normalization: rescale the pion field, $\vec{\pi} \rightarrow \vec{\pi}/F$

$$\mathcal{L}_{eff} = \frac{1}{2} \partial_\mu \vec{\pi} \cdot \partial^\mu \vec{\pi} + \frac{1}{48F^2} \text{tr}\{[\partial_\mu \pi, \pi] [\partial^\mu \pi, \pi]\} + \dots$$

- \Rightarrow a. Symmetry requires the pions to interact
- b. Derivative coupling: Goldstone bosons only interact if their momentum does not vanish λ/π^4

- Expression given for \mathcal{L}_{eff} only holds if the external fields are turned off. Also, $\text{tr}(\partial_\mu U \partial^\mu U^\dagger)$ is invariant only under global transformations

Suffices to replace $\partial_\mu U$ by

$$D_\mu U = \partial_\mu U - i(v_\mu + a_\mu)U + iU(v_\mu - a_\mu)$$

In contrast to $\text{tr}(\partial_\mu U \partial^\mu U^\dagger)$, the term $\text{tr}(D_\mu U D^\mu U^\dagger)$ is invariant under local $SU(2)_R \times SU(2)_L$

- Can construct further invariants: $s + ip$ transforms like $U \Rightarrow \text{tr}\{(s + ip)U^\dagger\}$ is invariant
Violates parity, but $\text{tr}\{(s + ip)U^\dagger\} + \text{tr}\{(s - ip)U\}$ is even under $p \rightarrow -p, \vec{\pi} \rightarrow -\vec{\pi}$

In addition, \exists invariant independent of U :

$$D_\mu \theta D^\mu \theta, \text{ with } D_\mu \theta = \partial_\mu \theta + 2 \text{tr}(a_\mu)$$

- Count the external fields as

$$\theta = O(1), \quad v_\mu, a_\mu = O(p), \quad s, p = O(p^2)$$

- Derivative expansion yields string of the form

$$\mathcal{L}_{eff} = \mathcal{L}^{(2)} + \mathcal{L}^{(4)} + \mathcal{L}^{(6)} + \dots$$

- Full expression for leading term:

$$\mathcal{L}^{(2)} = \frac{F^2}{4} \langle D_\mu U D^\mu U^\dagger + \chi U^\dagger + U \chi^\dagger \rangle + h_0 D_\mu \theta D^\mu \theta$$

$$\chi \equiv 2B(s + ip), \quad \langle X \rangle \equiv \text{tr}(X)$$

- Contains 3 constants: F, B, h_0
“effective coupling constants”
- Next-to-leading order:

$$\begin{aligned} \mathcal{L}^{(4)} = & \frac{\ell_1}{4} \langle D_\mu U D^\mu U \rangle^2 + \frac{\ell_2}{4} \langle D_\mu U D_\nu U \rangle \langle D^\mu U D^\nu U \rangle \\ & + \frac{\ell_3}{4} \langle \chi U^\dagger + U \chi^\dagger \rangle^2 + \frac{\ell_4}{4} \langle D_\mu \chi D^\mu U^\dagger + D_\mu U D^\mu \chi^\dagger \rangle \\ & + \dots \end{aligned}$$

- Number of effective coupling constants rapidly grows with the order of the expansion

- Infinitely many effective coupling constants
Symmetry does not determine these
Predictivity ?
- Essential point: If \mathcal{L}_{eff} is known to given order
 \Rightarrow can work out low energy expansion of the
Green functions to that order (Weinberg 1979)
- NLO expressions for F_π, M_π involve 2 new
coupling constants: l_3, l_4 .

In the $\pi\pi$ scattering amplitude, two further
coupling constants enter at NLO: l_1, l_2 .

- Note: effective theory is a quantum field theory
Need to perform the path integral

$$e^{iS_{\text{QCD}}\{v,a,s,p,\theta\}} = \mathcal{N}_{eff} \int [d\pi] e^{i \int dx \mathcal{L}_{eff}\{\vec{\pi},v,a,s,p,\theta\}}$$

- Classical theory \Leftrightarrow tree graphs
Need to include graphs with loops
- Power counting in dimensional regularization:
Graphs with ℓ loops are suppressed by factor $p^{2\ell}$ as compared to tree graphs
- \Rightarrow Leading contributions given by tree graphs
Graphs with one loop contribute at next-to-leading order, etc.
- The leading contribution to S_{QCD} is given by the sum of all tree graphs = classical action:

$$S_{\text{QCD}}\{v, a, s, p, \theta\} = \text{extremum}_{U(x)} \int dx \mathcal{L}_{\text{eff}}\{U, v, a, s, p, \theta\}$$

III. Illustrations

12. Some tree level calculations

A. Condensate in terms of QCD

- To calculate the quark condensate of the massless theory, it suffices to consider the effective action for $v = a = p = \theta = 0$ and to take a constant scalar external field

$$s = \begin{pmatrix} m_u & 0 \\ 0 & m_d \end{pmatrix}$$

- Expansion in powers of m_u and m_d treats

$$H_1 = \int d^3x \{m_u \bar{u}u + m_d \bar{d}d\} \text{ as a perturbation}$$

$$S_{\text{QCD}}\{0, 0, m, 0, 0\} = S_{\text{QCD}}^0 + S_{\text{QCD}}^1 + \dots$$

- S_{QCD}^0 is independent of the quark masses (cosmological constant)
- S_{QCD}^1 is linear in the quark masses

- First order in $m_u, m_d \Rightarrow$ expectation value of H_1 in unperturbed ground state is relevant

$$S_{\text{QCD}}^1 = - \int dx \langle 0 | m_u \bar{u}u + m_d \bar{d}d | 0 \rangle$$

- \Rightarrow $\langle 0 | \bar{u}u | 0 \rangle$ and $\langle 0 | \bar{d}d | 0 \rangle$ are the coefficients of the terms in S_{QCD} that are linear in m_u and m_d

B. Condensate in terms of effective theory

- Need the effective action for $v = a = p = \theta = 0$ to first order in s

\Rightarrow classical level of effective theory suffices.

- extremum of the classical action: $U = 1$

$$S_{\text{QCD}}^1 = \int dx F^2 B (m_u + m_d)$$

- comparison with

$$S_{\text{QCD}}^1 = - \int dx \langle 0 | m_u \bar{u}u + m_d \bar{d}d | 0 \rangle \text{ yields}$$

$$\boxed{\langle 0 | \bar{u}u | 0 \rangle = \langle 0 | \bar{d}d | 0 \rangle = -F^2 B} \quad (1)$$

C. Evaluation of M_π at tree level

- In classical theory, the square of the mass is the coefficient of the term in the Lagrangian that is quadratic in the meson field:

$$\begin{aligned}\frac{F^2}{4}\langle\chi U^\dagger + U\chi^\dagger\rangle &= \frac{F^2 B}{2}\langle m(U^\dagger + U)\rangle \\ &= F^2 B(m_u + m_d)\left\{1 - \frac{\vec{\pi}^2}{2F^2} + \dots\right\}\end{aligned}$$

Hence $\boxed{M_\pi^2 = (m_u + m_d)B}$ (2)

- Tree level result for F_π :

$$\boxed{F_\pi = F}$$
 (3)

- (1) + (2) + (3) \Rightarrow GMOR relation:

$$\boxed{M_\pi^2 = \frac{(m_u + m_d) |\langle 0 | \bar{u}u | 0 \rangle|}{F_\pi^2}}$$

13. M_π beyond tree level

- The formula $M_\pi^2 = (m_u + m_d)B$ only holds at tree level, represents leading term in expansion of M_π^2 in powers of m_u, m_d
- Disregard isospin breaking: set $m_u = m_d = m$

A. M_π to 1 loop

- Claim: at next-to-leading order, the expansion of M_π^2 in powers of m contains a logarithm:

$$M_\pi^2 = M^2 - \frac{1}{2} \frac{M^4}{(4\pi F)^2} \ln \frac{\Lambda_3^2}{M^2} + O(M^6)$$

$$M^2 \equiv 2mB$$

- Proof: Pion mass \Leftrightarrow pole position, for instance in the Fourier transform of $\langle 0 | T A_a^\mu(x) A_b^\nu(y) | 0 \rangle$
Suffices to work out the perturbation series for this object to one loop of the effective theory

- Result

$$M_\pi^2 = M^2 + \frac{2\ell_3 M^4}{F^2} + \frac{M^2}{2F^2} \frac{1}{i} \Delta(0, M^2) + O(M^6)$$

$\Delta(0, M^2)$ is the propagator at the origin

$$\begin{aligned} \Delta(0, M^2) &= \frac{1}{(2\pi)^d} \int \frac{d^d p}{M^2 - p^2 - i\epsilon} \\ &= i(4\pi)^{-d/2} \Gamma(1 - d/2) M^{d-2} \end{aligned}$$

- Contains a pole at $d = 4$:

$$\Gamma(1 - d/2) = \frac{2}{d - 4} + \dots$$

- Divergent part is proportional to M^2 :

$$\begin{aligned} M^{d-2} &= M^2 \mu^{d-4} (M/\mu)^{d-4} = M^2 \mu^{d-4} e^{(d-4) \ln(M/\mu)} \\ &= M^2 \mu^{d-4} \{1 + (d-4) \ln(M/\mu) + \dots\} \end{aligned}$$

- Denote the singular factor by

$$\begin{aligned} \lambda &\equiv \frac{1}{2} (4\pi)^{-d/2} \Gamma(1 - d/2) \mu^{d-4} \\ &= \frac{\mu^{d-4}}{16\pi^2} \left\{ \frac{1}{d-4} - \frac{1}{2} (\ln 4\pi + \Gamma'(1) + 1) + O(d-4) \right\} \end{aligned}$$

- The propagator at the origin then becomes

$$\frac{1}{i}\Delta(0, M^2) = M^2 \left\{ 2\lambda + \frac{1}{16\pi^2} \ln \frac{M^2}{\mu^2} + O(d-4) \right\}$$

- In the expression for M_π^2

$$M_\pi^2 = M^2 + \frac{2\ell_3 M^4}{F^2} + \frac{M^2}{2F^2} \frac{1}{i}\Delta(0, M^2) + O(M^6)$$

the divergence can be absorbed in ℓ_3 :

$$\ell_3 = -\frac{1}{2}\lambda + \ell_3^{\text{ren}}$$

- ℓ_3^{ren} depends on the renormalization scale μ

$$\ell_3^{\text{ren}} = \frac{1}{64\pi^2} \ln \frac{\mu^2}{\Lambda_3^2} \quad \text{running coupling constant}$$

- Λ_3 is the ren. group invariant scale of ℓ_3

Net result for M_π^2

$$M_\pi^2 = M^2 - \frac{1}{2} \frac{M^4}{(4\pi F)^2} \ln \frac{\Lambda_3^2}{M^2} + O(M^6)$$

$\Rightarrow M_\pi^2$ contains a chiral logarithm at NLO

- Crude estimate for Λ_3 , based on SU(3) mass formulae for the pseudoscalar octet:

$$0.2 \text{ GeV} < \Lambda_3 < 2 \text{ GeV}$$

$$\bar{\ell}_3 \equiv \ln \frac{\Lambda_3^2}{M_\pi^2} = 2.9 \pm 2.4$$

Gasser & L. 1984

⇒ Next-to-leading term is small correction:

$$0.005 < \frac{1}{2} \frac{M_\pi^2}{(4\pi F_\pi)^2} \ln \frac{\Lambda_3^2}{M_\pi^2} < 0.04$$

- Scale of the expansion is set by size of pion mass in units of decay constant:

$$\frac{M^2}{(4\pi F)^2} \simeq \frac{M_\pi^2}{(4\pi F_\pi)^2} = 0.0144$$

B. M_π to 2 loops

- Terms of order m_{quark}^3 :

$$M_\pi^2 = M^2 - \frac{1}{2} \frac{M^4}{(4\pi F)^2} \ln \frac{\Lambda_3^2}{M^2} + \frac{17}{18} \frac{M^6}{(4\pi F)^4} \left(\ln \frac{\Lambda_M^2}{M^2} \right)^2 + k_M M^6 + O(M^8)$$

F is pion decay constant for $m_u = m_d = 0$

ChPT to two loops

Colangelo 1995

- Coefficients $\frac{1}{2}$ and $\frac{17}{18}$ determined by symmetry
- Λ_3, Λ_M and $k_M \iff$ coupling constants in \mathcal{L}_{eff}

14. F_π to one loop

- Also contains a logarithm at NLO:

$$F_\pi = F \left\{ 1 - \frac{M^2}{16\pi^2 F^2} \ln \frac{M^2}{\Lambda_4^2} + O(M^4) \right\}$$
$$M_\pi^2 = M^2 \left\{ 1 + \frac{M^2}{32\pi^2 F^2} \ln \frac{M^2}{\Lambda_3^2} + O(M^4) \right\}$$

F is pion decay constant in limit $m_u, m_d \rightarrow 0$

- Structure is the same, coefficients and scale of logarithm are different

- Low energy theorem: Λ_4 also determines the slope of the scalar form factor to leading order

$$\langle r^2 \rangle_s = \frac{6}{(4\pi F)^2} \left\{ \ln \frac{\Lambda_4^2}{M_\pi^2} - \frac{13}{12} + O(M^2) \right\}$$

- Scalar form factor of the pion can be calculated by means of dispersion theory
- Result for the slope:

$$\langle r^2 \rangle_s = 0.61 \pm 0.04 \text{ fm}^2$$

Colangelo, Gasser & L. Nucl. Phys. 2001

⇒ Corresponding value of the scale Λ_4 :

$$\Lambda_4 = 1.26 \pm 0.14 \text{ GeV}$$

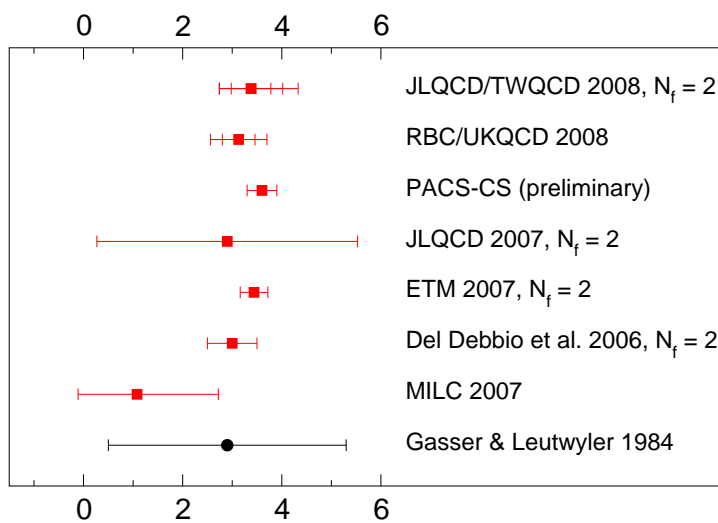
15. Lattice results for M_π, F_π

A. Results for M_π

- Determine the scale Λ_3 by comparing the lattice results for M_π as function of m with the χ PT formula

$$M_\pi^2 = M^2 - \frac{1}{2} \frac{M^4}{(4\pi F)^2} \ln \frac{\Lambda_3^2}{M^2} + O(M^6)$$

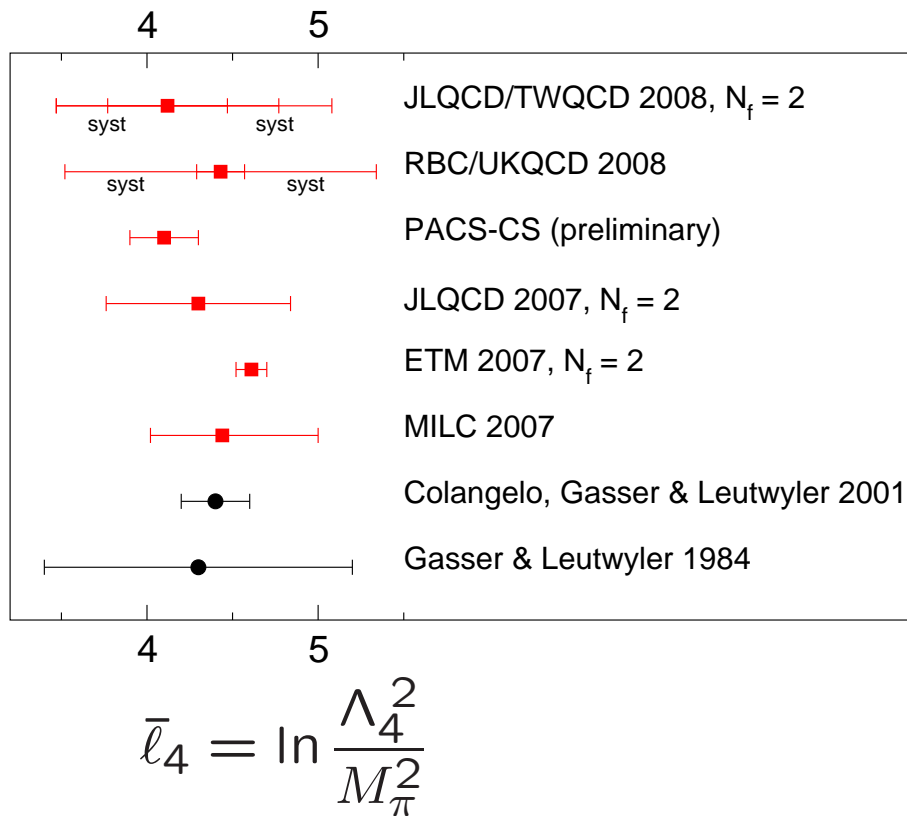
$$M^2 \equiv 2Bm$$



Horizontal axis shows the value of $\bar{l}_3 \equiv \ln \frac{\Lambda_3^2}{M_\pi^2}$

Range for Λ_3 obtained in 1984 corresponds to $\bar{l}_3 = 2.9 \pm 2.4$

Result of RBC/UKQCD 2008: $\bar{l}_3 = 3.13 \pm 0.33_{stat} \pm 0.24_{syst}$



- Lattice results beautifully confirm the prediction for the sensitivity of F_π to m_u, m_d :

$$\frac{F_\pi}{F} = 1.072 \pm 0.007 \quad \text{Colangelo and Dürr 2004}$$

16. $\pi\pi$ scattering

A. Low energy scattering of pions

- Consider scattering of pions with $\vec{p} = 0$
 - At $\vec{p} = 0$, only the S-waves survive (angular momentum barrier). Moreover, these reduce to the scattering lengths
 - Bose statistics: S-waves cannot have $I = 1$, either have $I = 0$ or $I = 2$
- \Rightarrow At $\vec{p} = 0$, the $\pi\pi$ scattering amplitude is characterized by two constants: a_0^0, a_0^2
- Chiral symmetry suppresses the interaction at low energy: Goldstone bosons of zero momentum do not interact
- $\Rightarrow a_0^0, a_0^2$ disappear in the limit $m_u, m_d \rightarrow 0$
- $\Rightarrow a_0^0, a_0^2 \sim M_\pi^2$ measure symmetry breaking

B. Tree level of χ PT

- Low Energy theorem Weinberg 1966:

$$a_0^0 = \frac{7M_\pi^2}{32\pi F_\pi^2} + O(M_\pi^4)$$

$$a_0^2 = -\frac{M_\pi^2}{16\pi F_\pi^2} + O(M_\pi^4)$$

\Rightarrow Chiral symmetry predicts a_0^0, a_0^2 in terms of F_π

- Accuracy is limited: Low energy theorem only specifies the first term in the expansion in powers of the quark masses
Corrections from higher orders ?

C. Scattering lengths at 1 loop

- Next term in the chiral perturbation series:

$$a_0^0 = \frac{7M_\pi^2}{32\pi F_\pi^2} \left\{ 1 + \frac{9}{2} \frac{M_\pi^2}{(4\pi F_\pi)^2} \ln \frac{\Lambda_0^2}{M_\pi^2} + O(M_\pi^4) \right\}$$

- Coefficient of chiral logarithm unusually large
Strong, attractive final state interaction
- Scale Λ_0 is determined by the coupling constants of $\mathcal{L}_{eff}^{(4)}$:

$$\frac{9}{2} \ln \frac{\Lambda_0^2}{M_\pi^2} = \frac{20}{21} \bar{\ell}_1 + \frac{40}{21} \bar{\ell}_2 - \frac{5}{14} \bar{\ell}_3 + 2\bar{\ell}_4 + \frac{5}{2}$$

- Information about $\bar{\ell}_1, \dots, \bar{\ell}_4$?

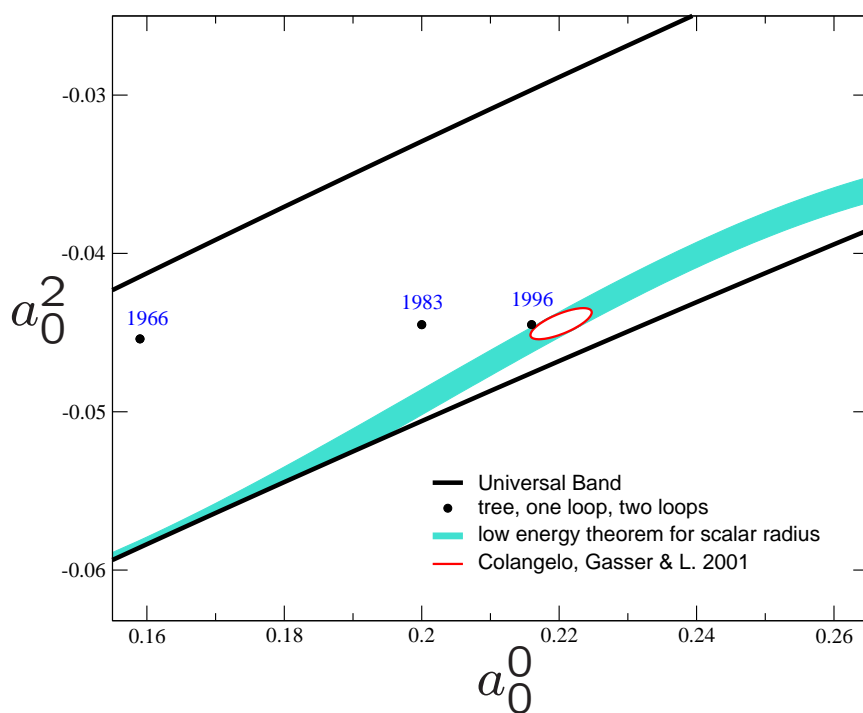
$\bar{\ell}_1, \bar{\ell}_2 \iff$	momentum dependence of scattering amplitude
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\Rightarrow Can be determined phenomenologically

$\bar{\ell}_3, \bar{\ell}_4 \iff$	dependence of scattering amplitude on quark masses
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Have discussed their values already

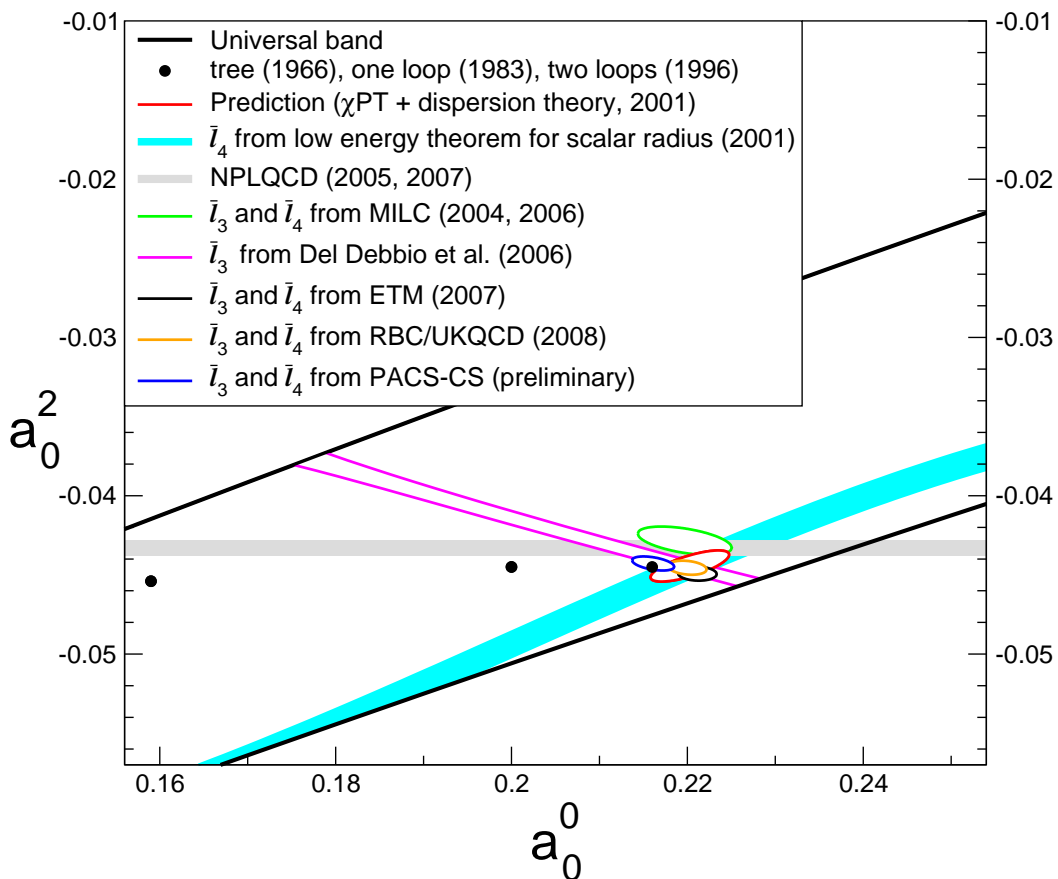
D. Numerical predictions from χ PT



Sizable corrections in a_0^0
 a_0^2 nearly stays put

E. Consequence of lattice results for l_3, l_4

- Uncertainty in prediction for a_0^0, a_0^2 is dominated by the uncertainty in the effective coupling constants l_3, l_4
- Can make use of the lattice results for these



F. Experiments concerning a_0^0, a_0^2

- Production experiments $\pi N \rightarrow \pi\pi N$,
 $\psi \rightarrow \pi\pi\omega$, $B \rightarrow D\pi\pi$, . . .

Problem: pions are not produced in vacuo

⇒ Extraction of $\pi\pi$ scattering amplitude is not simple

Accuracy rather limited

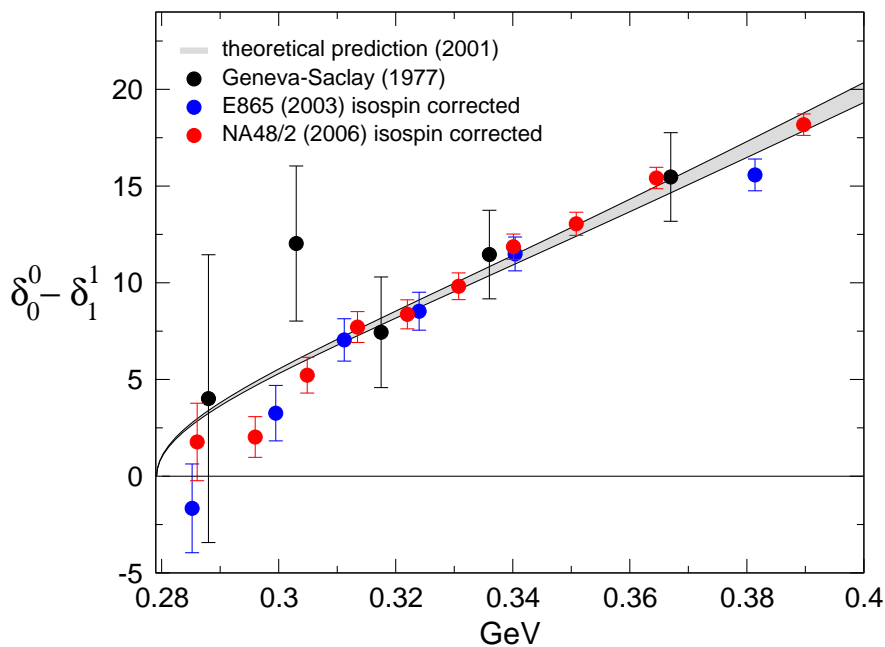
- $K^\pm \rightarrow \pi^+\pi^-e^\pm\nu$ data:
CERN-Saclay, E865, NA48/2
- $K^\pm \rightarrow \pi^0\pi^0\pi^\pm$, $K^0 \rightarrow \pi^0\pi^0\pi^0$: cusp near threshold, NA48/2
- $\pi^+\pi^-$ atoms, DIRAC

G. Results from K_{e4} decay

$$K^\pm \rightarrow \pi^+ \pi^- e^\pm \nu$$

- Allows clean measurement of $\delta_0^0 - \delta_1^1$

Theory predicts $\delta_0^0 - \delta_1^1$ as function of energy



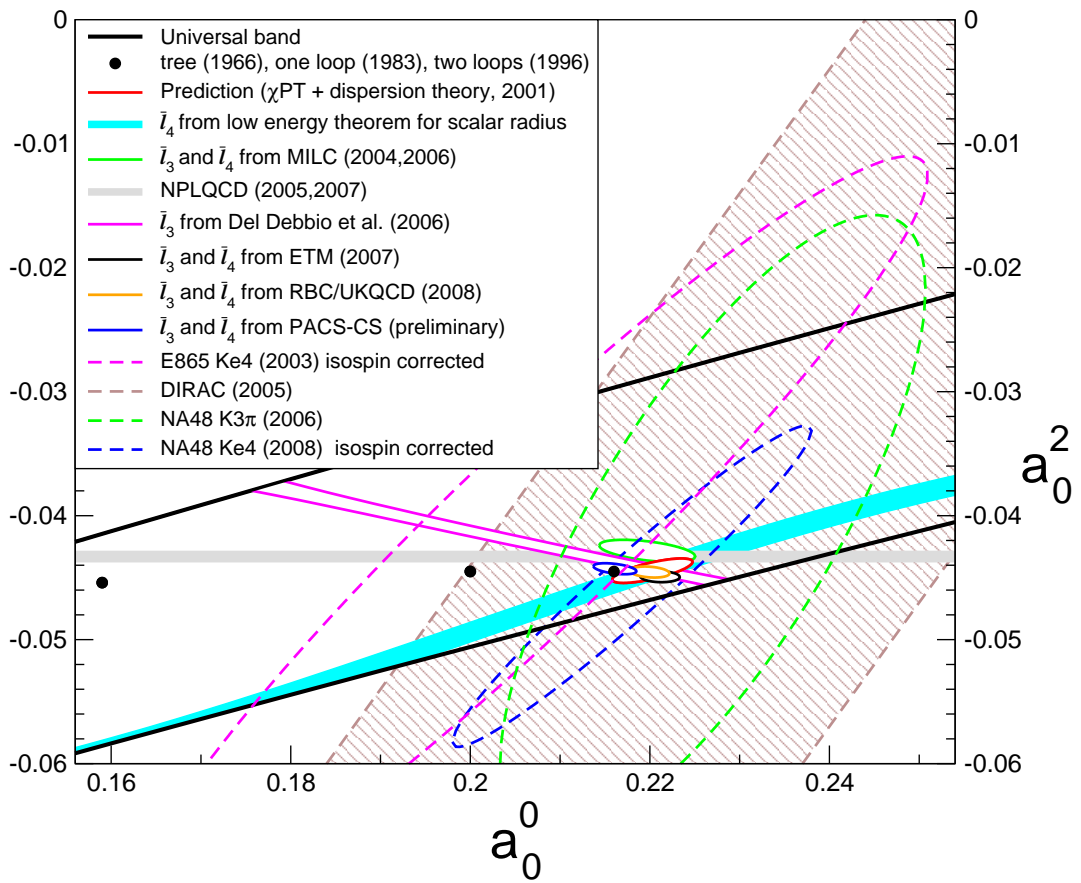
There was a discrepancy here, because a pronounced isospin breaking effect from

$$K \rightarrow \pi^0 \pi^0 e \nu \rightarrow \pi^+ \pi^- e \nu$$

had not been accounted for in the data analysis

⇒ Colangelo, Gasser, Rusetsky, arXiv:0811.0775

H. Summary for a_0^0, a_0^2



17. Conclusions for $SU(2) \times SU(2)$

- Expansion in powers of m_u, m_d yields a very accurate low energy representation of QCD
- Lattice results confirm the GMOR relation
- ⇒ M_π is dominated by the contribution from the quark condensate
- ⇒ Energy gap of QCD is understood very well
- Lattice approach allows an accurate measurement of the effective coupling constant l_3 already now
- Even for l_4 , the lattice starts becoming competitive with dispersion theory

18. Expansion in powers of m_s

- Theoretical reasoning
 - The eightfold way is an approximate symmetry
 - The only coherent way to understand this within QCD: $m_s - m_d$, $m_d - m_u$ can be treated as perturbations
 - Since $m_u, m_d \ll m_s$
 - ⇒ m_s can be treated as a perturbation
 - ⇒ Expect expansion in powers of m_s to work, but convergence to be comparatively slow
- This can now also be checked on the lattice

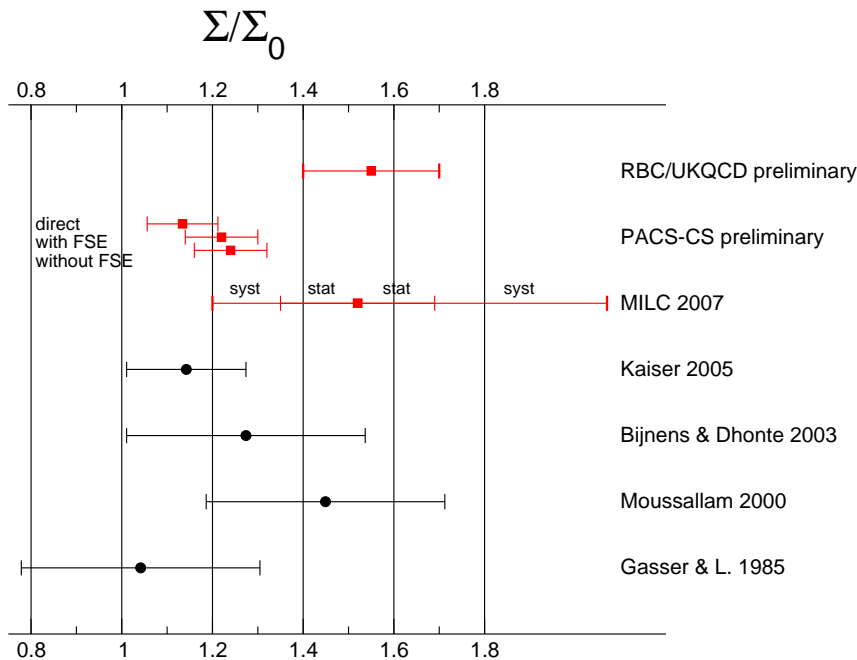
- Consider the limit $m_u, m_d \rightarrow 0$, m_s physical
 - F is value of F_π in this limit
 - Σ is value of $|\langle 0 | \bar{u}u | 0 \rangle|$ in this limit
 - B is value of $M_\pi^2 / (m_u + m_d)$ in this limit
- Exact relation: $\Sigma = F^2 B$
- F_0, B_0, Σ_0 : values for $m_u = m_d = m_s = 0$
- *Paramagnetic inequalities*: both F and Σ should decrease if m_s is taken smaller

$$F > F_0, \Sigma > \Sigma_0$$

Jan Stern et al. 2000

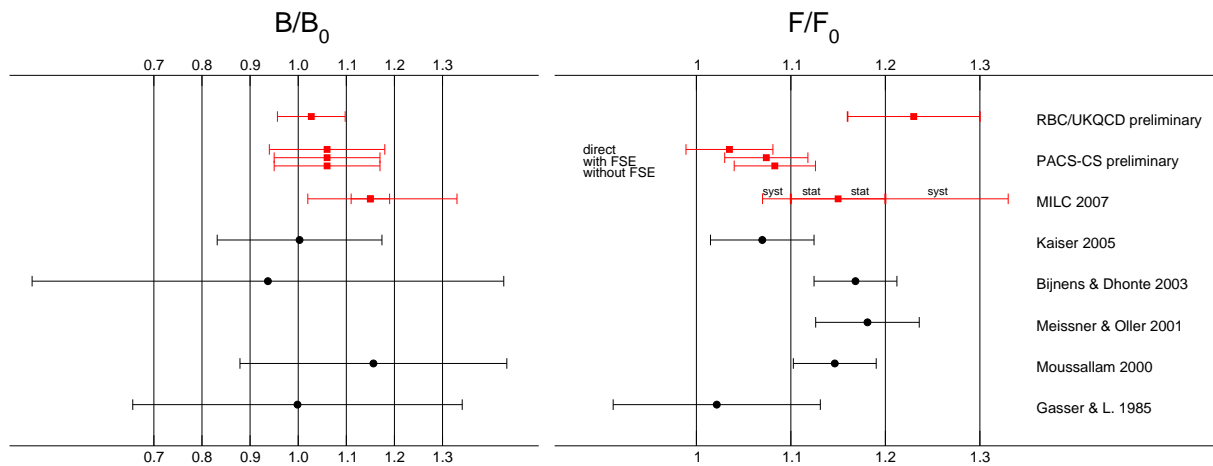
- $N_c \rightarrow \infty$: F, Σ, B become independent of m_s
- $\Rightarrow (F/F_0 - 1), (\Sigma/\Sigma_0 - 1), (B/B_0 - 1)$
violate the OZI rule

A. Condensate



- Central values of RBC/UKQCD and PACS-CS for Σ/Σ_0 lead to qualitatively different conclusions concerning OZI-violations
- ⇒ Discrepancy indicates large systematic errors
- The lattice results confirm the parametric inequalities, but do not yet allow to draw conclusions about the size of the OZI-violations

B. Results for B , F



- Results for B are coherent, indicate small OZI-violations in B
- $\Rightarrow F$ is the crucial factor in $\Sigma = F^2 B$
- Note: most of the numbers quoted are preliminary, errors purely statistical, continuum limit, finite size effects, ...

C. Expansion to NLO

Involves the effective coupling constants L_4 and L_6 of the $SU(3) \times SU(3)$ Lagrangian:

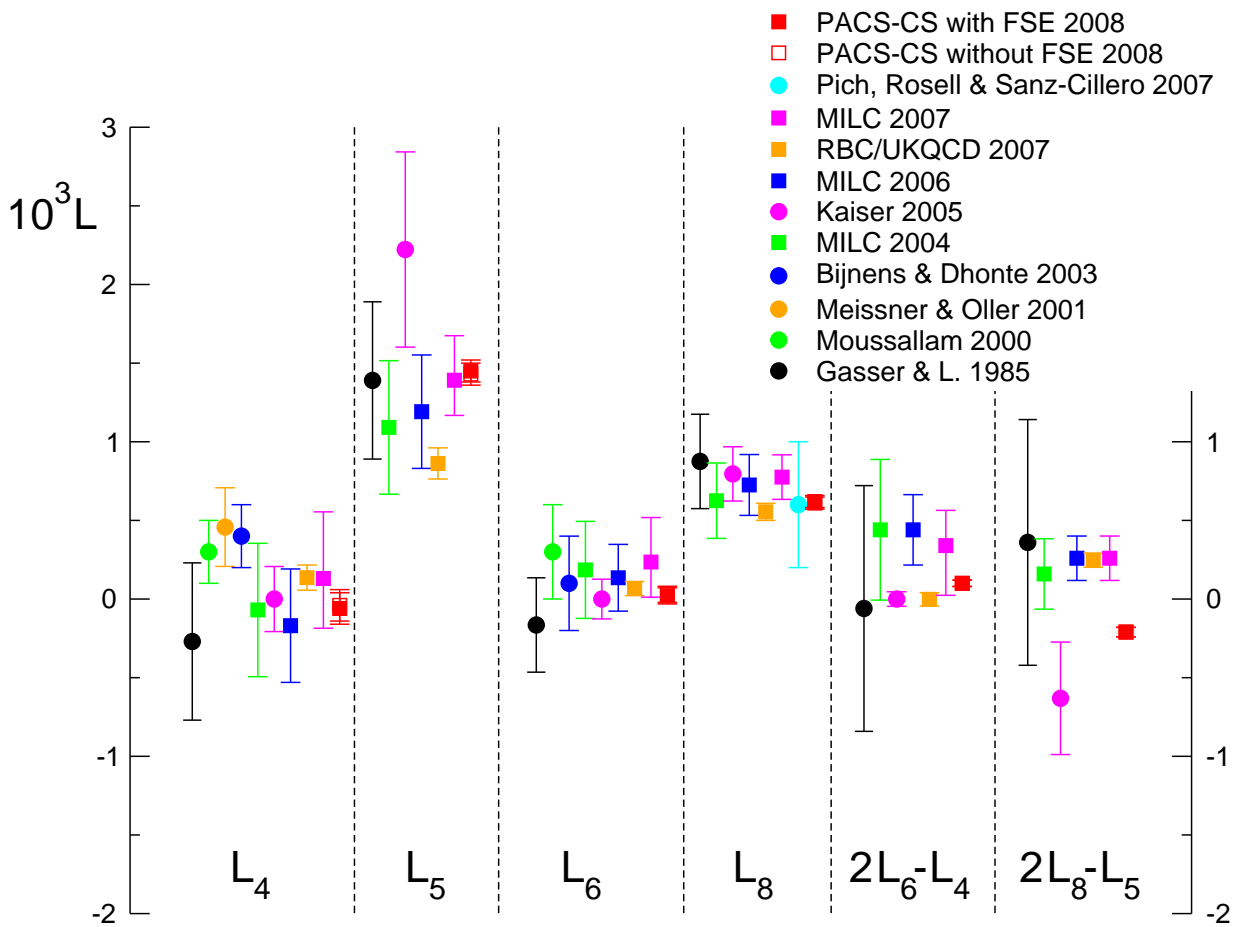
$$F/F_0 = 1 + \frac{8\bar{M}_K^2}{F_0^2} L_4 + \chi \log + \dots$$

$$\Sigma/\Sigma_0 = 1 + \frac{32\bar{M}_K^2}{F_0^2} L_6 + \chi \log + \dots$$

$$B/B_0 = 1 + \frac{16\bar{M}_K^2}{F_0^2} (2L_6 - L_4) + \chi \log + \dots$$

\bar{M}_K is the kaon mass for $m_u = m_d = 0$.

D. Running coupling constants L_4, L_5, L_6, L_8



Numerical values shown refer to running scale $\mu = M_\rho$

19. Conclusions for $SU(3) \times SU(3)$

- In $B/B_0 \leftrightarrow 2L_6 - L_4$, the available lattice results indicate little if any violations of the OZI rule
- For F/F_0 (and hence Σ/Σ_0), the situation is not conclusive: some of the data indicate very juicy OZI-violations, others are consistent with $F/F_0 \simeq \Sigma/\Sigma_0 \simeq 1$
- If the central value $F/F_0 = 1.23$ of RBC/UKQCD were confirmed within small uncertainties, we would be faced with a qualitative puzzle:
 - F_π is the pion wave function at the origin
 - F_K is larger because one of the two valence quarks is heavier \rightarrow moves more slowly \rightarrow wave function more narrow \rightarrow higher at the origin: $F_K/F_\pi \simeq 1.19$
 - $F/F_0 = 1.23$ indicates that the wave function is more sensitive to the mass of the sea quarks than to the mass of the valence quarks . . . very strange \rightarrow most interesting if true

- The PACS-CS results are consistent with our old estimates. Only show modest violations of the OZI rule. If these results are confirmed, then the picture looks very coherent, also for $SU(3) \times SU(3)$.

20. Puzzling results on $K_L \rightarrow \pi \mu \nu$

- Hadronic matrix element of weak current:

$$\langle K^0 | \bar{u} \gamma^\mu s | \pi^- \rangle = (p_K + p_\pi)^\mu f_+(t) + (p_K - p_\pi)^\mu f_-(t)$$

- Scalar form factor $\sim \langle K^0 | \partial_\mu (\bar{u} \gamma^\mu s) | \pi^- \rangle$

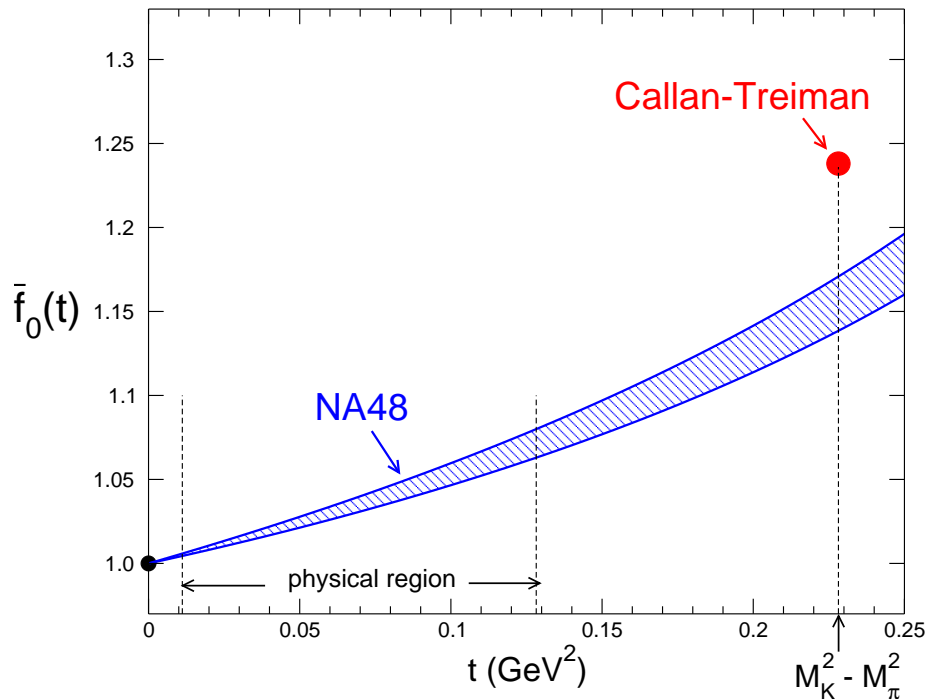
$$f_0(t) = f_+(t) + \frac{t}{M_K^2 - M_\pi^2} f_-(t)$$

- Low energy theorem Callan & Treiman 1966

$$f_0(M_K^2 - M_\pi^2) = \frac{F_K}{F_\pi} \left\{ 1 + O(m_u, m_d) \right\} \simeq 1.19$$

$f_0(0) = f_+(0) \simeq 0.96$ relevant for
determination of V_{us}

- Comparison with experiment



NA48, Phys. Lett. B647 (2007) 341 (141 authors, 2.3×10^6 events)

- Plot shows normalized scalar form factor

$$\bar{f}_0(t) = \frac{f_0(t)}{f_0(0)}$$

- CT relation in this normalization:

$$\bar{f}_0(M_K^2 - M_\pi^2) = \frac{F_K}{F_\pi f_+(0)} = 1.2446 \pm 0.0041$$

Bernard and Passemar 2008

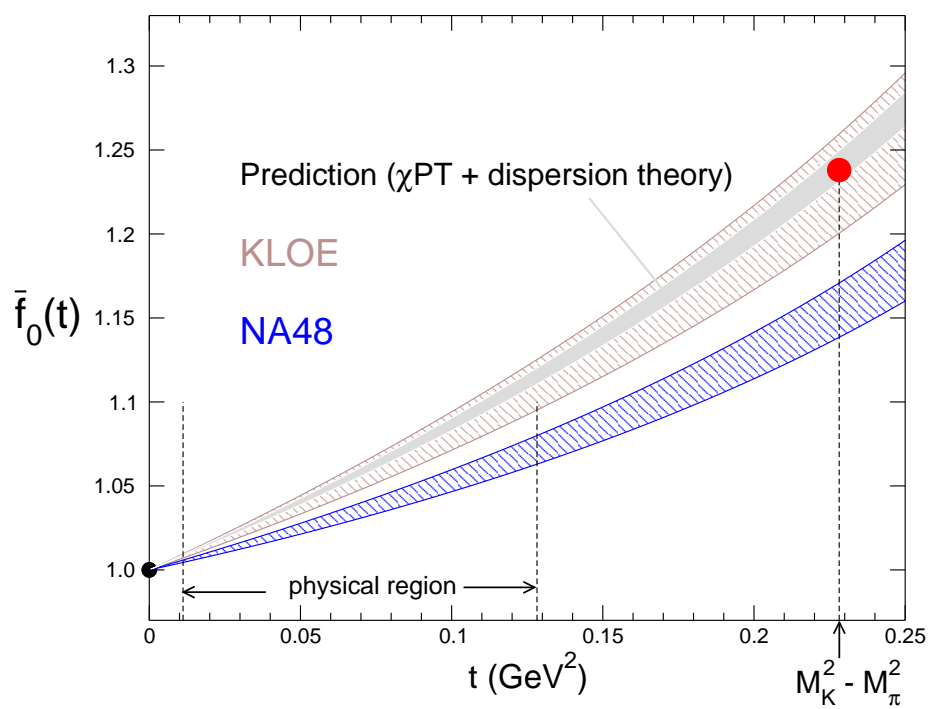
- Implications

- NA48 data on $K_L \rightarrow \pi\mu\nu$ disagree with SM
- If confirmed, the implications are dramatic:
⇒ W couples also to right-handed currents

Bernard, Oertel, Passemar & Stern 2006

- There are not many places where the SM disagrees with observation, need to investigate these carefully
- At low energies, high precision is required

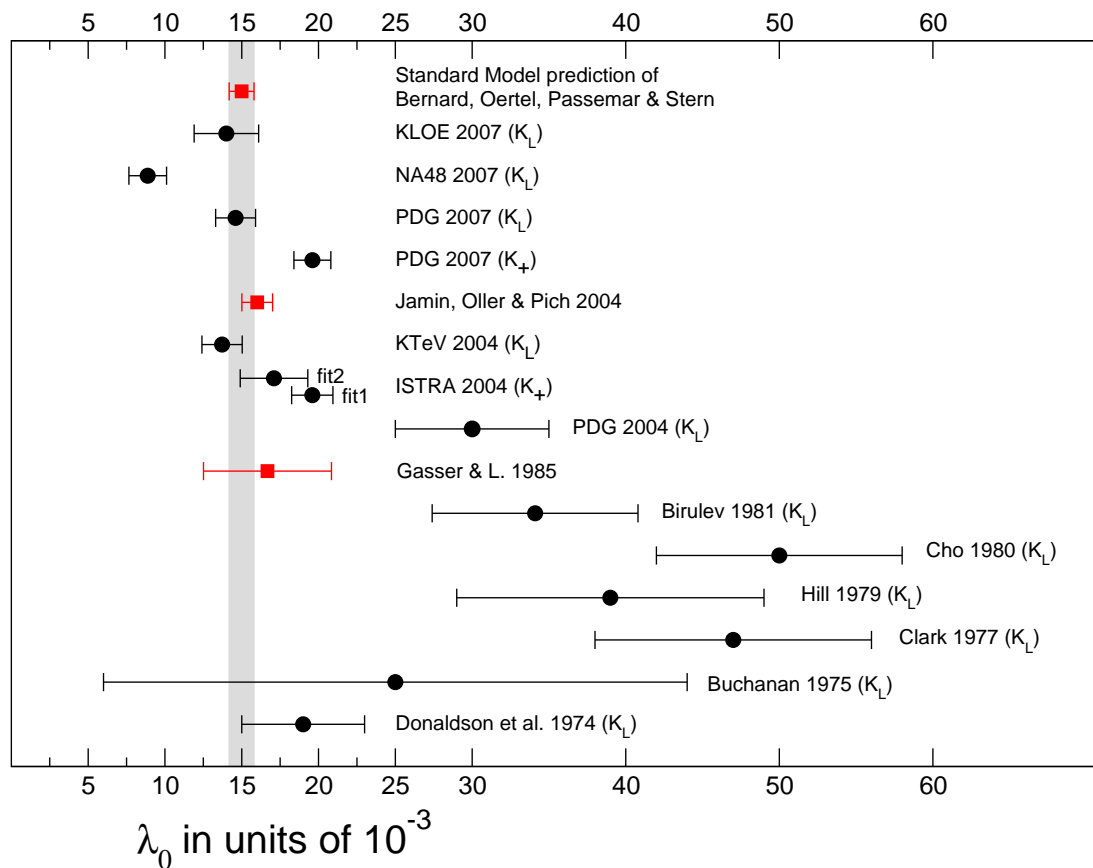
- New data from KLOE



I thank Emilie Passemar for some of the material shown in this figure

- History of the issue: data on the slope of the scalar form factor

$$f_0(t) = f_0(0) \left\{ 1 + \frac{\lambda_0 t}{M_{\pi^+}^2} + O(t^2) \right\}$$



Exercises

1. Evaluate the positive frequency part of the massless propagator

$$\Delta^+(z, 0) = \frac{i}{(2\pi)^3} \int \frac{d^3k}{2k^0} e^{-ikz}, \quad k^0 = |\vec{k}|$$

for $\text{Im } z^0 < 0$. Show that the result can be represented as

$$\Delta^+(z, 0) = \frac{1}{4\pi i z^2}$$

2. Evaluate the d -dimensional propagator

$$\Delta(z, M) = \int \frac{d^d k}{(2\pi)^d} \frac{e^{-ikz}}{M^2 - k^2 - i\epsilon}$$

at the origin and verify the representation

$$\Delta(0, M) = \frac{i}{4\pi} \Gamma\left(1 - \frac{d}{2}\right) \left(\frac{M^2}{4\pi}\right)^{\frac{d}{2}-1}$$

How does this expression behave when $d \rightarrow 4$?

3. Leading order effective Lagrangian:

$$\begin{aligned}\mathcal{L}^{(2)} &= \frac{F^2}{4} \langle D_\mu U D^\mu U^\dagger + \chi U^\dagger + U \chi^\dagger \rangle + h_0 D_\mu \theta D^\mu \theta \\ D_\mu U &= \partial_\mu U - i(v_\mu + a_\mu)U + iU(v_\mu - a_\mu) \\ \chi &= 2B(s + ip) \\ D_\mu \theta &= \partial_\mu \theta + 2\langle a_\mu \rangle \\ \langle X \rangle &= \text{tr} X\end{aligned}$$

- Take the space-time independent part of the external field $s(x)$ to be isospin symmetric (i. e. set $m_u = m_d = m$):

$$s(x) = m \mathbf{1} + \tilde{s}(x)$$

- Expand $U = \exp i\phi/F$ in powers of $\phi = \vec{\phi} \cdot \vec{\tau}$ and check that, in this normalization of the field ϕ , the kinetic part takes the standard form

$$\mathcal{L}^{(2)} = \frac{1}{2} \partial_\mu \vec{\phi} \cdot \partial^\mu \vec{\phi} - \frac{1}{2} M^2 \vec{\phi}^2 + \dots$$

with $M^2 = 2mB$.

- Draw the graphs for all of the interaction vertices containing up to four of the fields $\phi, v_\mu, a_\mu, \tilde{s}, p, \theta$.

4. Show that the classical field theory belonging to the QCD Lagrangian in the presence of external fields is invariant under

$$\begin{aligned}
v'_\mu + a'_\mu &= V_R(v_\mu + a_\mu)V_R^\dagger - i\partial_\mu V_R V_R^\dagger \\
v'_\mu - a'_\mu &= V_L(v_\mu - a_\mu)V_L^\dagger - i\partial_\mu V_L V_L^\dagger \\
s' + ip' &= V_R(s + ip)V_L^\dagger \\
q'_R &= V_R q_R(x) \\
q'_L &= V_L q_L
\end{aligned}$$

where V_R, V_L are space-time dependent elements of $U(3)$.

5. Evaluate the pion mass to NLO of χ PT. Draw the relevant graphs and verify the representation

$$M_\pi^2 = M^2 + \frac{2\ell_3 M^4}{F^2} + \frac{M^2}{2F^2} \frac{1}{i} \Delta(0, M^2) + O(M^6)$$

6. Start from the symmetry property of the effective action,

$$S_{\text{QCD}}\{v', a', s', p', \theta'\} = S_{\text{QCD}}\{v, a, s, p, \theta\} - \int dx \langle \beta \Omega \rangle,$$

and show that this relation in particular implies the Ward identity

$$\begin{aligned}
\partial_\mu^x \langle 0 | T A_a^\mu(x) P_b(y) | 0 \rangle &= -\frac{1}{4} i \delta(x-y) \langle 0 | \bar{q} \{ \lambda_a, \lambda_b \} q | 0 \rangle \\
&\quad + \langle 0 | T \bar{q}(x) i \gamma_5 \{ m, \frac{1}{2} \lambda_a \} q(x) P_b(y) | 0 \rangle \\
a &= 1, \dots, 8, \quad b = 0, \dots, 8
\end{aligned}$$

7. What is the Ward identity obeyed by the singlet axial current,

$$\partial_\mu^x \langle 0 | T A_0^\mu(x) P_b(y) | 0 \rangle = ?$$