Nonassociative Quantum Field Theory

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January 9, 2008

Overview

- A Philosophical Approach to the Fundamental Equations of Nature
- Nonassociative Field Theory and Octonion Calculus
- Nonassociative Quantum Field Theory and Path Integral Quantization
- Technological Implications: Design Ideas for Interesting Devices

(1) A Philosophical Approach to the Fundamental Equations of Nature

Is there a fundamental theory that gives a unified description of nature?

The best experimentally verified approximation today:

- The quantum field theory of the standard model of elementary particle physics
  - Description of all forces except gravity
- The classical theory of general relativity
  - Different approaches to quantum gravity do not integrate naturally with the standard model

Mathematical beauty as guide in the unification process

The idea of *mathematical beauty* proofed to be an effective guide in the unification process.

The prime examples are:

- Maxwell's equations
- Electroweak unification in the standard model

Bottom-up approach

The strategy to unify the mathematical description of experimentally derived physical models can be seen as an *bottom-up* approach.

Define the set of *all possible* sets of model equations $\mathcal{M}$. With the initial physical theory $\tau_0 \in \mathcal{M}$ given, the unification step to a more condensed theory is a map $\varphi : \mathcal{M} \to \mathcal{M}$ with

$$\tau_0 \mapsto \varphi(\tau_0) =: \tau_1 \tag{1}$$

The process towards a unified theory can be seen as the sequence

$$(\tau_0, \tau_1, \ldots, \tau_N) =: (\tau_n)_{n=0}^{N} \tag{2}$$

The limit theory τ^*

Is there a limit theory?

$$\lim_{n\to\infty} \tau_n = \tau^* \tag{3}$$

Is the process of unification of experimentally derived physical theories rich enough to generate a sequence that converges to the unified theory?

- Is the model space $\mathcal{M}$ defined with todays mathematics rich enough?
- Can we get to the unified theory in finite steps?
- Can we accelerate the convergence?
- Can we even compute the limit?

Unreasonable effectiveness of mathematics in natural sciences

All these questions are hard to answer. What we definitely know is that the unification approach worked up to the standard model.

- What can we gain from the concept of the limit theory τ^*?
- Is this concept of any use to identify the unified theory?

Top-down approach

The key observation is that we can reverse the point of view and directly look at the limit theory τ^* and use a *top-down* approach to identify the mathematical structure of the limit theory.

The question now is:

- What is the underlying mathematical structure of τ^*?

The identification is greatly simplified by the observation that all experimentally derived physical theories must be special cases of τ^*. This greatly restricts the possible underlying mathematical structure we want to identify.

Implications of the existence of τ^*

Assume:

- *The limit theory* τ^* *exists.*

If the reduction process to identify the unified theory has a limit, then by construction the limit theory must contain simple mathematical structures.

Completeness of $\mathcal{M}$ with respect to the limit theory

Assume:

- *The model space $\mathcal{M}$ is complete.*

If $\mathcal{M}$ is complete the limit theory τ^* can be represented within todays mathematics. The underlying mathematical structure can thus be expressed with the tools of mathematics.

Conceptual power of the approach

If we assume existence of τ^* and completeness of $\mathcal{M}$ then the *top-down* approach to identify the unified theory is an extremely powerful concept.

Even if we can identify only part of the fundamental structure, the formal mathematical method enables the *systematic* construction of derived theories.

These can be used to model at least certain aspects of physical processes and clarify deeper symmetries.

Heuristic approach to the identification of the Cayley algebra

The proposed top-down approach suggests to look for common features in well established physical model equations.

We can identify:

- Field theories over the reals $\mathbb{R}$ and complex numbers $\mathbb{C}$.
  - Heat equation
  - Schrödinger's equation
- Hidden field theories over the quaternions $\mathbb{H}$
  - Maxwell's equations
  - Dirac's equation

Normed division algebras and the Cayley algebra

What is the unifying principle behind $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$*?*

All three are normed division algebras and there exists only one more: The Cayley algebra also known as octonions $\mathbb{O}$.

- Octonions are nonassociative!

The goal is set:

Construct a field theory over the octonions!

(2) Nonassociative Field Theory and Octonion Calculus

We want to investigate a classical field theory over the octonions, but there is a problem:

Not associative? No calculus!

We immediately hit a serious road block. The octonions are typically used only in an algebra setting.

A closer look at the octonions is required!

Division algebras

For the following a vector space A will be a finite-dimensional module over the field of the real numbers. An algebra is a vector space A together with a bilinear map $\phi : A \times A \mapsto A$, called *multiplication* and a nonzero element $1 \in A$ called the *unit* such that

$$\forall a \in A : \phi(1, a) = \phi(a, 1) = 1 \tag{4}$$

In the usual way we abbreviate the multiplication

$$ab := \phi(a, b) \tag{5}$$

An algebra A is a division algebra when

$$\forall a, b \in A : ab = 0 \Rightarrow \; a = 0 \vee b = 0 \tag{6}$$

A normed division algebra is an algebra A that is a normed vector space with

$$\forall a, b \in A : \|ab\| = \|a\|\|b\| \tag{7}$$

The octonions are not associative but *alternative* with the following definition:

$$\forall a, b \in A : (aa)b = a(ab),\ (ab)a = a(ba),\ (ba)a = b(aa) \tag{8}$$

Multiplication table and basic properties

e_0	e_1	e_2	e_3	e_4	e_5	e_6	e_7
e_1	$-e_0$	$-e_3$	e_2	$-e_5$	e_4	e_7	$-e_6$
e_2	e_3	$-e_0$	$-e_1$	$-e_6$	$-e_7$	e_4	e_5
e_3	$-e_2$	e_1	$-e_0$	$-e_7$	e_6	$-e_5$	e_4
e_4	e_5	e_6	e_7	$-e_0$	$-e_1$	$-e_2$	$-e_3$
e_5	$-e_4$	e_7	$-e_6$	e_1	$-e_0$	e_3	$-e_2$
e_6	$-e_7$	$-e_4$	e_5	e_2	$-e_3$	$-e_0$	e_1
e_7	e_6	$-e_5$	$-e_4$	e_3	e_2	$-e_1$	$-e_0$

Table 1: Multiplication table for the Cayley numbers

The simplest way to construct the octonions is to give their multiplication table. The octonions are an eigth-dimensional algebra with basis elements $e_0, e_1, e_2, e_3, e_4, e_5, e_6, e_7$. The multiplication table describes the result of multiplying the element in the ith row by the element in the jth column.

The multiplication table reveals the following properties:

- e_0 is the unit.
- $e_1, \ldots, e_7$ are square roots of -1.
- e_i and e_j anticommute

$$e_i e_j = -e_j e_i, \quad i, j \in \{1, \ldots, 7\}, \quad i \neq j \tag{9}$$

While the above properties are true for all possible multiplication tables implementing the Cayley numbers, here we have included some additional symmetries:

- e_0 spans the subalgebra of the real numbers $\mathbb{R}$

- e_0, e_1 span the subalgebra of the complex numbers $\mathbb{C}$
- e_0, e_1, e_2, e_3 span the subalgebra of the quaternions $\mathbb{H}$

Cayley-Dickson construction

The symmetries of the multiplication table reflect the Cayley-Dickson doubling process used for the construction of the octonions. A complex number $a + ib$ can be thought as a pair of real numbers (a, b) with the multiplication defined as

$$(a, b)(c, d) := (ac - db, ad + cb) \tag{10}$$

The complex conjugation of a complex number can be defined by

$$(a, b)^* := (a, -b) \tag{11}$$

Then the strategy from above can be extended to the quaternions as pairs of complex numbers by slightly modifying the multiplication.

$$(a, b)(c, d) := (ac - db^*, a^*d + cb) \tag{12}$$

The conjugation for the quaternions is defined as

$$(a, b)^* := (a^*, -b) \tag{13}$$

The game continues and we can define the octonions as pairs of quaternions with the multiplication law from above.

$$(a, b)(c, d) := (ac - db^*, a^*d + cb) \tag{14}$$

Note that now the order of the factors on the right hand side is important, since the quaternions do not commute.

Matrix representations

Now we want to understand the the multiplication in $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$, and $\mathbb{O}$ as matrix operations. The basis elements of the complex numbers can be represented as

$$1 \sim \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad i \sim \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \tag{15}$$

A complex number is thus given by

$$a + ib \sim \begin{pmatrix} a & -b \\ b & a \end{pmatrix} \tag{16}$$

The basis elements of the quaternions have also matrix representations.

$$1 \sim \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{17}$$

$$i \sim \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \sim \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix} \tag{18}$$

$$j \sim \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \sim \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \tag{19}$$

$$k \sim \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} \sim \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \tag{20}$$

A quaternion can be represented in matrix form in the following way:

$$a + ib + jc + kd \sim \begin{pmatrix} a + ib & -c + id \\ c + id & a - ib \end{pmatrix} \sim \begin{pmatrix} a & -b & -c & -d \\ b & a & d & -c \\ c & -d & a & b \\ d & c & -b & a \end{pmatrix} \tag{21}$$

Left and right actions

Due to the nonassociativity the octonions cannot be directly represented by matrices. Matrix algebras are intrinsically associative due to the ordinary matrix multiplication. One way to capture the essence of the octonion multiplication is to look at left and right actions. Define the bracket operation $[\cdot] : \mathbb{O} \to \mathbb{R}^8$ extracting the coefficients of an octonion.

$$[x] = \left[\sum_{n=0}^{7} x_n e_n\right] := \begin{pmatrix} x_0 \\ x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \\ x_7 \end{pmatrix} \tag{22}$$

With this notation the left action and the right action are defined as

$$[ax] := L_a[x] \tag{23}$$

$$[xa] := R_a[x] \tag{24}$$

The key observation now is that the algebra of the left and right actions is associative.

Matrix representation of the left action

$$L_{e_0} \sim \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \tag{25}$$

$$L_{e_1} \sim \begin{pmatrix} i & 0 & 0 & 0 \\ 0 & -i & 0 & 0 \\ 0 & 0 & -i & 0 \\ 0 & 0 & 0 & i \end{pmatrix} \sim \begin{pmatrix} 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix} \tag{26}$$

$$L_{e_2} \sim \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix} \sim \begin{pmatrix} 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \end{pmatrix} \tag{27}$$

$$L_{e_3} \sim \begin{pmatrix} 0 & i & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & 0 & -i \\ 0 & 0 & -i & 0 \end{pmatrix} \sim \begin{pmatrix} 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \end{pmatrix} \tag{28}$$

$$L_{e_4} \sim \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \sim \begin{pmatrix} 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \end{pmatrix} \tag{29}$$

$$L_{e_5} \sim \begin{pmatrix} 0 & 0 & i & 0 \\ 0 & 0 & 0 & i \\ i & 0 & 0 & 0 \\ 0 & i & 0 & 0 \end{pmatrix} \sim \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \tag{30}$$

$$L_{e_6} \sim \begin{pmatrix} 0 & 0 & 0 & -1^* \\ 0 & 0 & 1^* & 0 \\ 0 & -1^* & 0 & 0 \\ 1^* & 0 & 0 & 0 \end{pmatrix} \sim \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \tag{31}$$

$$L_{e_7} \sim \begin{pmatrix} 0 & 0 & 0 & -i^* \\ 0 & 0 & i^* & 0 \\ 0 & -i^* & 0 & 0 \\ i^* & 0 & 0 & 0 \end{pmatrix} \sim \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \tag{32}$$

The complex basis elements 1^*, i^* are defined as follows:

$$1^* \sim \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad i^* \sim \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \tag{33}$$

Left-associative octonion algebra

We now focus on pure left actions and analyze the transformation properties of the left actions. There are four sets of matrices:

$$\mathcal{S}_0 := \{L_{e_0}\} \tag{34}$$

$$\mathcal{S}_1 := \{L_{e_i} : \quad 1 \le i \le 7\} \tag{35}$$

$$\mathcal{S}_2 := \{L_{e_i}L_{e_j} : \quad 1 \le i < j \le 7\} \tag{36}$$

$$\mathcal{S}_3 := \{L_{e_i}L_{e_j}L_{e_k} : \quad 1 \le i < j < k \le 7\} \tag{37}$$

The cardinalities of these sets of matrices are:

$$\#\mathcal{S}_0 = 1, \quad \#\mathcal{S}_1 = 7, \quad \#\mathcal{S}_2 = 21, \quad \#\mathcal{S}_3 = 35 \tag{38}$$

Define the union

$$\mathcal{S} := \mathcal{S}_0 \cup \mathcal{S}_1 \cup \mathcal{S}_2 \cup \mathcal{S}_3 \tag{39}$$

then the sum of all cardinalities $1 + 7 + 21 + 35 = 64$ is equal to the dimension of the vector space spanned by all real 8×8 matrices. Thus we have a complete basis set. The basis elements above are an orthonormal system with respect to the scalar product:

$$(A, B) := \frac{1}{8}\mathrm{tr}(A^T B) \tag{40}$$

The link to quantum theory: Dirac's equation

Dirac's equation can be embedded in a scalar equation in the octonion context. The Dirac matrices can be identified as octonion left-actions. A four component complex bispinor can canonically identified with an eight-dimensional vector and thus represented as octonion.

$$i\partial_0[\psi(x)] = \sum_{n=1}^{7} L_{e_n}\partial_n[\psi(x)] \tag{41}$$

The octonion $x = \sum_{n=0}^{7} e_n x_n$ represents the time coordinate x_0 the space coordinates x_1, x_2, x_3 and four additional dimensions x_4, x_5, x_6, x_7. Restriction to spacetime coordinates and derivatives gives the standard Dirac equation. The mass term can be generated by a phase factor in one of the additional dimensions.

Spin operators in higher dimensions

The spin operators are proportional to the product of two Dirac matrices and represent rotations on the bispinor. Define the generator of a rotation in the (k, l) plane as

$$S_{k,l} := \frac{1}{2} L_{e_k} L_{e_l}, \quad k, l \in \{1, \dots, 7\}, k \neq l \tag{42}$$

with the usual transformation law

$$[\psi'] := \exp(S_{k,l}\omega)[\psi] \tag{43}$$

In the space dimensions we get the usual spin operators here with the complex unit absorbed in the definition. Only the matrices $L_{e_0}, \dots, L_{e_5}$ are complex representable not the matrices L_{e_6}, L_{e_7}. What is the spin operator for the plane $(6, 7)$?

$$S_{6,7} := \frac{1}{2} L_{e_6} L_{e_7} = \frac{i}{2} \mathbf{1}_4 \tag{44}$$

The transformation generated is a global gauge transformation!

$$[\psi'] := \exp(i\mathbf{1}_4\omega/2)[\psi] \tag{45}$$

Gauge transformations are geometric rotations in the extended space. This relation cannot be expressed in the standard framework of quantum mechanics, because L_{e_6}, L_{e_7} are not complex representable!

What is the spin operator for the plane $(5,4)$?

$$S_{5,4} := \frac{1}{2}L_{e_5}L_{e_4} = \frac{i}{2}\gamma_5 \tag{46}$$

The γ_5 matrix appears with the transformation

$$[\psi'] := \exp(i\gamma_5\omega/2)[\psi] \tag{47}$$

To complete the list of diagonal generators we calculate the spin operator for the plane $(3,2)$.

$$S_{3,2} := \frac{1}{2} L_{e_3} L_{e_2} = \frac{i}{2} \Sigma_3 \tag{48}$$

The automorphism group of the Cayley algebra: G_2

The automorphism group of the octonions is the exceptional Lie group G_2. This group leaves the octonion multiplication invariant. The group $G_2 \subset SO(7)$ contains a subset of seven-dimensional rotations. Essentially the group elements of G_2 are simultaneous rotations in two planes of $\mathbb{R}^7$.

The spin operators above do not generate G_2 elements, only special linear combinations do.

$$H_1 := S_{3,2} - S_{6,7} = \frac{i}{2}(\Sigma_3 - \mathbf{1}_4) \tag{49}$$

$$H_2 := S_{6,7} - S_{5,4} = \frac{i}{2}(\mathbf{1}_4 - \gamma_5) \tag{50}$$

$$H_3 := S_{5,4} - S_{3,2} = \frac{i}{2}(\gamma_5 - \Sigma_3) \tag{51}$$

The generators H_1, H_2, H_3 are linear dependent. Two of them form the Cartan subalgebra of the Lie algebra $\mathfrak{g}_2$. The significance here is that the projector

$(\mathbf{1}_4 - \gamma_5)$ that is closely related to the electroweak interaction. The generator of the electromagnetic interaction is supplemented with the spin operator Σ_3.

The Cartan algebra is key to the classification of states in different representations. Aside from the Cartan algebra there are twelve generators left in $\mathfrak{g}_2$. There are two more generators W_1, W_2 aside from H_2, that act entirely in the subspace spanned by x_4, x_5, x_6, x_7. These three form a $\mathfrak{su}(2)$ subalgebra of $\mathfrak{g}_2$.

Since all generators are in the adjoint representation of G_2 we identify H_1 with the photon and H_2, W_1, W_2 as the gauge bosons of the weak interaction. The fixed algebraic relation between H_1 and H_2 predicts a Weinberg angle of $\theta = 30^\circ$.

With the identification of the unique structure of the electroweak interaction physical quantum numbers, the electric charge and weak isospin, can be associated with the eigenvalues, the roots, of the Cartan algebra.

Root diagrams for the representations of G_2 in the Cayley algebra

The left actions generate four sets of matrices that are invariant under the action of G_2.

$$\mathcal{S}_0 := \{L_{e_0}\} \tag{52}$$

$$\mathcal{S}_1 := \{L_{e_i} : \quad 1 \leq i \leq 7\} \tag{53}$$

$$\mathcal{S}_2 := \{L_{e_i} L_{e_j} : \quad 1 \leq i < j \leq 7\} \tag{54}$$

$$\mathcal{S}_3 := \{L_{e_i} L_{e_j} L_{e_k} : \quad 1 \leq i < j < k \leq 7\} \tag{55}$$

These sets decompose further under the action of G_2 and we get:

$$\mathcal{S}_0 = \mathbf{1}_0, \quad \mathcal{S}_1 = \mathbf{7}_1, \quad \mathcal{S}_2 = \mathbf{7}_2 + \mathbf{14}_2, \quad \mathcal{S}_3 = \mathbf{1}_3 + \mathbf{7}_3 + \mathbf{27}_3 \tag{56}$$

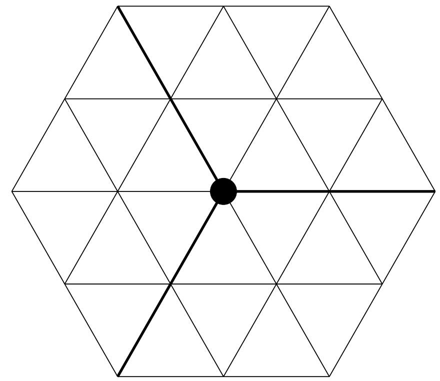


Figure 1: Trivial representation: $\mathbf{1}$

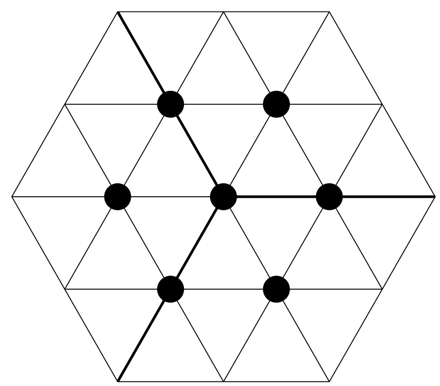


Figure 2: Fundamental representation: $\mathbf{7}$

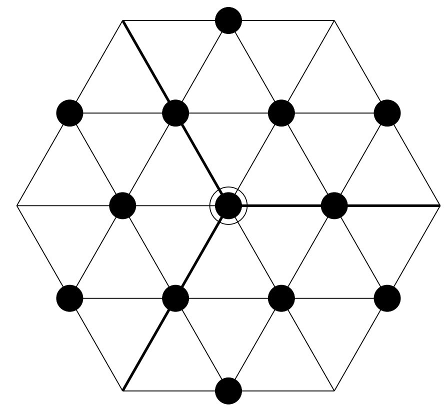


Figure 3: Adjoint representation: **14**

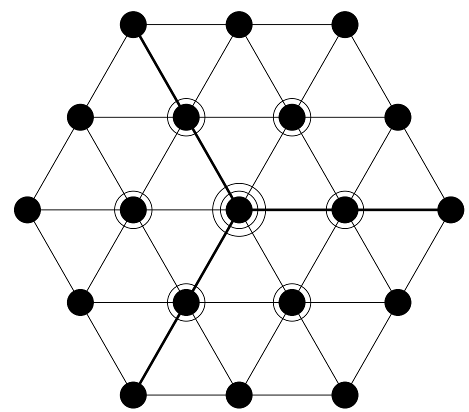


Figure 4: Huge representation: **27**

These representations can be identified with the vacuum $\mathbf{1}_0$, the leptons $\mathbf{7}_1$, the mesons $\mathbf{7}_2$, the gauge bosons $\mathbf{14}_2$ the antisymmetric baryon $\mathbf{1}_3$, the nucleons $\mathbf{7}_3$ and symmetric baryons $\mathbf{27}_3$.

Cartan-Weyl classification

The classification of the states using the Cartan-Weyl approach requires the complex extension of the matrix algebra $M_8(\mathbb{R})$ to $M_8(\mathbb{C})$. The roots for the state $\Phi \in M_8(\mathbb{C})$ are defined as

$$[H_n, \Phi] = i\alpha_n \Phi, \quad n \in \{1, 2, 3\} \tag{57}$$

with the identification $(\alpha_1, \alpha_2, \alpha_3) = (Q_e, Q_w, Q_m)$ and Q_e the electric charge Q_w the weak isospin and $Q_e + Q_w + Q_m = 0$.

Leptons

The classification scheme predicts a neutral lepton state, identified as right-handed neutrino. The representations of G_2 can be further classified with respect to the $SU(3) \subset G_2$ subgroup. We introduce for convenience a color notation. The color singlet $\mathbf{1}$ is defined as

$$\nu_R^0 = A = (0, 0, 0) \tag{58}$$

The fundamental representation $\mathbf{3}$ with respect to the $SU(3)$ symmetry reads

$$e_R^+ = R = (1, 0, -1), \quad \nu_L^+ = G = (0, -1, 1), \quad e_L^- = B = (-1, 1, 0) \tag{59}$$

and for the complex conjugate representation $\bar{\mathbf{3}}$ we have

$$e_R^- = \bar{R} = (-1, 0, 1), \quad \nu_L^- = \bar{G} = (0, 1, -1), \quad e_L^+ = \bar{B} = (1, -1, 0) \tag{60}$$

Gauge-Boson States

The true physical states are linear combinations of the states calculated above. The bracket denotes the commutator.

$$i\frac{1}{\sqrt{2}}([B,\bar{B}]-[R,\bar{R}]) \tag{61}$$

$$\frac{1}{\sqrt{2}}([B,\bar{G}]+[\bar{B},G]), \quad i\frac{1}{\sqrt{2}}([B,\bar{G}]-[\bar{B},G]) \tag{62}$$

$$\frac{1}{\sqrt{2}}([R,\bar{B}]+[\bar{R},B]), \quad i\frac{1}{\sqrt{2}}([R,\bar{B}]-[\bar{R},B]) \tag{63}$$

$$\frac{1}{\sqrt{2}}([G,\bar{R}]+[\bar{G},R]), \quad i\frac{1}{\sqrt{2}}([G,\bar{R}]-[\bar{G},R]) \tag{64}$$

$$i\frac{1}{\sqrt{6}}([B,\bar{B}]-2[G,\bar{G}]+[R,\bar{R}]) \tag{65}$$

$$\frac{1}{\sqrt{6}}([B,G]+[\bar{B},\bar{G}])+i\frac{1}{\sqrt{3}}([R,A]-[\bar{R},A]) \tag{66}$$

$$i\frac{1}{\sqrt{6}}([B,G]-[\bar{B},\bar{G}])+\frac{1}{\sqrt{3}}([R,A]+[\bar{R},A]) \tag{67}$$

$$\frac{1}{\sqrt{6}}([R,B]+[\bar{R},\bar{B}])+i\frac{1}{\sqrt{3}}([G,A]-[\bar{G},A]) \tag{68}$$

$$i\frac{1}{\sqrt{6}}([R,B]-[\bar{R},\bar{B}])+\frac{1}{\sqrt{3}}([G,A]+[\bar{G},A]) \tag{69}$$

$$\frac{1}{\sqrt{6}}([G,R]+[\bar{G},\bar{R}])+i\frac{1}{\sqrt{3}}([B,A]-[\bar{B},A]) \tag{70}$$

$$i\frac{1}{\sqrt{6}}([G,R]-[\bar{G},\bar{R}])+\frac{1}{\sqrt{3}}([B,A]+[\bar{B},A]) \tag{71}$$

Meson States

The meson states are orthogonal to the gauge-boson states.

$$i\frac{1}{\sqrt{3}}([B,\bar{B}]+[G,\bar{G}]+[R,\bar{R}]) \tag{72}$$

$$\frac{1}{\sqrt{3}}([B,G]+[\bar{B},\bar{G}])-i\frac{1}{\sqrt{6}}([R,A]-[\bar{R},A]) \tag{73}$$

$$-i\frac{1}{\sqrt{3}}([B,G]-[\bar{B},\bar{G}])+\frac{1}{\sqrt{6}}([R,A]+[\bar{R},A]) \tag{74}$$

$$\frac{1}{\sqrt{3}}([R,B]+[\bar{R},\bar{B}])-i\frac{1}{\sqrt{6}}([G,A]-[\bar{G},A]) \tag{75}$$

$$-i\frac{1}{\sqrt{3}}([R,B]-[\bar{R},\bar{B}])+\frac{1}{\sqrt{6}}([G,A]+[\bar{G},A]) \tag{76}$$

$$\frac{1}{\sqrt{3}}([G,R]+[\bar{G},\bar{R}])-i\frac{1}{\sqrt{6}}([B,A]-[\bar{B},A]) \tag{77}$$

$$-i\frac{1}{\sqrt{3}}([G,R]-[\bar{G},\bar{R}])+\frac{1}{\sqrt{6}}([B,A]+[\bar{B},A]) \tag{78}$$

Baryon Singlet State

The totally antisymmetric baryon singlet state is a nontrivial linear combination. The bracket denotes the mixed commutator anti-commutator $[A, B, C] := \{A, [B, C]\}$.

$$\sqrt{\frac{2}{7}}([R, G, B] + [\bar{R}, \bar{G}, \bar{B}]) + i\frac{1}{\sqrt{7}}([A, \bar{R}, R] + [A, \bar{G}, G] + [A, \bar{B}, B]) \qquad (79)$$

Baryon Septet States

The first state is neutral with quantum numbers $(0, 0)$.

$$i\frac{1}{\sqrt{2}}([R, G, B] - [\bar{R}, \bar{G}, \bar{B}]) \tag{80}$$

The next state has the quantum number $(-1, -1)$.

$$\frac{1}{\sqrt{2}}[A, \bar{R}, \bar{G}] + i\frac{1}{2}([B, \bar{R}, R] + [B, \bar{G}, G]) \tag{81}$$

The complete spectrum can be constructed by cyclic permutation of (R, G, B) of the last state and the corresponding anti particles. All in all seven states.

Baryon 27-Multiplet States

The multiplet contains three neutral states. The first two states are orthogonalized in the standard way.

$$i\frac{1}{\sqrt{2}}([A,\bar{R},R]+[A,\bar{G},G]) \tag{82}$$

$$i\frac{1}{\sqrt{6}}([A,\bar{R},R]+[A,\bar{G},G]-2[A,\bar{B},B]) \tag{83}$$

$$\sqrt{\frac{3}{14}}([R,G,B]+[\bar{R},\bar{G},\bar{B}])-i\frac{2}{\sqrt{21}}([A,\bar{R},R]+[A,\bar{G},G]+[A,\bar{B},B]) \tag{84}$$

The last state is the twin of the baryon singlet state. All three states have quantum numbers $(0,0)$. The next states are in complex form.

$$\frac{1}{\sqrt{2}}[A,\bar{R},\bar{G}] - i\frac{1}{2}([B,\bar{R},R] + [B,\bar{G},G]) \tag{85}$$

$$i\frac{1}{2}([B,\bar{R},R] - [B,\bar{G},G]) \tag{86}$$

Plus two cyclic permutations of (R, G, B) in the last two states and the anti particles. All in all twelve states. The quantum numbers for the last two states are $(-1, -1)$.

$$[R,G,\bar{B}], \quad [A,R,\bar{B}], \quad [R,\bar{G},\bar{B}], \quad [A,G,\bar{B}] \tag{87}$$

$$[A,R,\bar{G}], \quad [\bar{R},G,\bar{B}] \tag{88}$$

The last six states are complex and have to be duplicated with their anti particles. The quantum numbers are $(-2,-2),(-2,0),(-2,2),(-1,-3)$ and $(-1,3),(0,-4)$. This gives twelve states.

Where are the quarks? Strong interaction?

The detailed representation theory above indicates that there is no need for the quark concept, because the meson and baryon states arise purely because of the nonassociativity of the octonions.

No need for quarks!

The root diagrams of the baryon octet and decuplet can be embedded in the G_2 root diagrams and the states identified. The root vectors for the fundamental quark states add up to the G_2 root vectors.

Quark states are an unfortunate choice of basis vectors!

No need for confinement!

The quark picture

The quark concept can be embedded in the root diagrams of G_2 with the identification of the root vectors of the fundamental representation $\mathbf{3}$ and $\bar{\mathbf{3}}$.

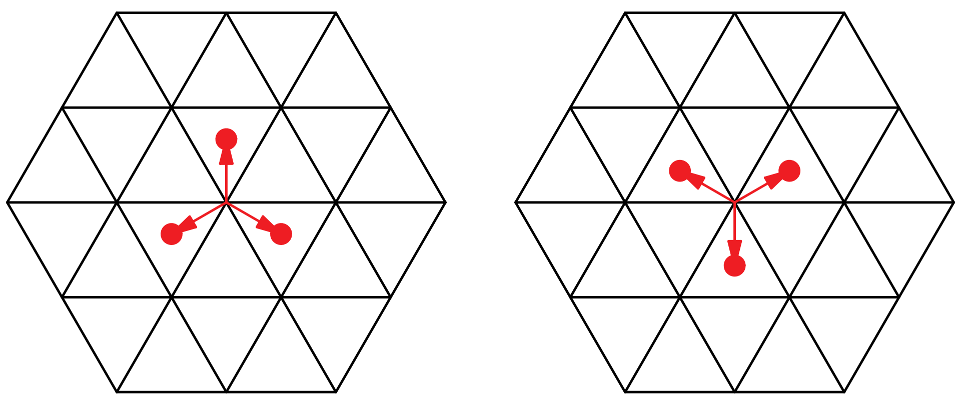


Figure 5: Fundamental Representation $\mathbf{3}$ and $\bar{\mathbf{3}}$ of $SU(3)$

The G_2 multiplet $\mathbf{27}_3$ decomposes into a baryon and antibaryon decuplet. The remaining states cannot be reached with three quarks except for the neutral state, but require five quarks: *Pentaquark* states!

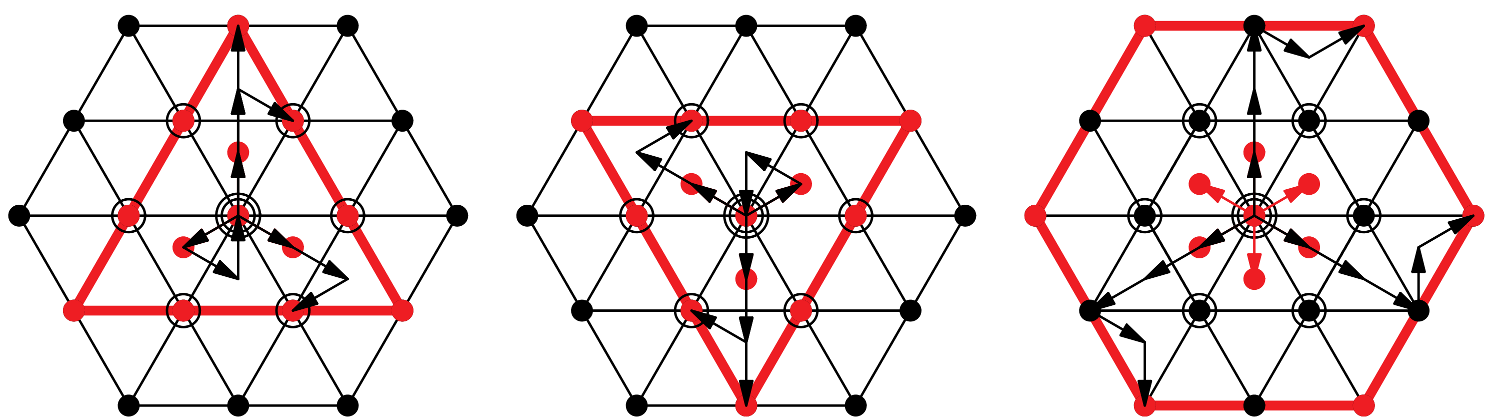


Figure 6: Quark embedding in $\mathbf{27}_3$

The G_2 multiplet $\mathbf{7}_3$ and $\mathbf{1}_3$ give the baryon octet of the nucleons. The G_2 multiplet $\mathbf{7}_2$ allows the embedding of meson states

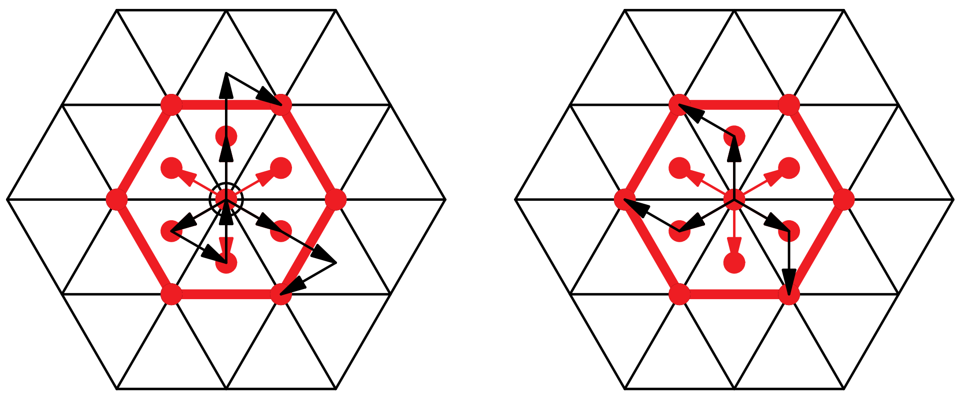


Figure 7: Quark embedding of nucleons and mesons

Since there are no quarks there is no gauge theory of the strong interaction.

- $SU(3)$ symmetry is present, but with respect to the electroweak interaction.
- The fundamental representation $\mathbf{3}$ and $\bar{\mathbf{3}}$ are lepton states!
- $SU(3) \supset U(1) \otimes SU(2)$ completion gives vector bosons with Higgs-like interaction terms

Gravity

The six vector bosons that are left, couple the right-handed neutrino ν_R^0 with the other leptons.

Universal interaction with all leptons! Gravity?!

Aside from the photon, that acts via the spin operator on the configuration space $(x_1, x_2, x_3) \in \mathbb{R}^3$, there are two vector bosons in this group that act via the other two spin components on this space.

The other four vector bosons act on the coordinates $(x_4, x_5, x_6, x_7) \in \mathbb{R}^4$ in the higher dimensional space.

(3) Nonassociative Quantum Field Theory and Path Integral Quantization

The classical field theory based on the generalization of the Dirac Lagrangian $\mathcal{L} := \mathrm{tr}(\Psi^* \partial \Psi)$ to higher dimensions can be quantized via the path integral method.

Work the nonabelian gauge theory program with gauge group G_2.

Problem: The gauge transformations have now a geometric meaning.

The idea of the covariant derivative and the minimal coupling are now geometric.

The gauge transformations are geometric rotations and thus coordinate transformations. This is the idea of general relativity! Invariance under general coordinate transformations.

Quantize the metric connection.

Conceptual fusion with general relativity!

Current research!

Feynman diagrams: They must be there anyway

Even if the details of the program outlined above are not clear. The representation theoretic decomposition implies a nontrivial structure in the Feynman vertex diagrams.

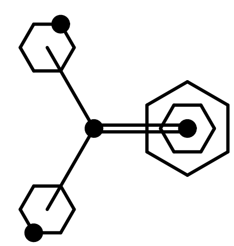


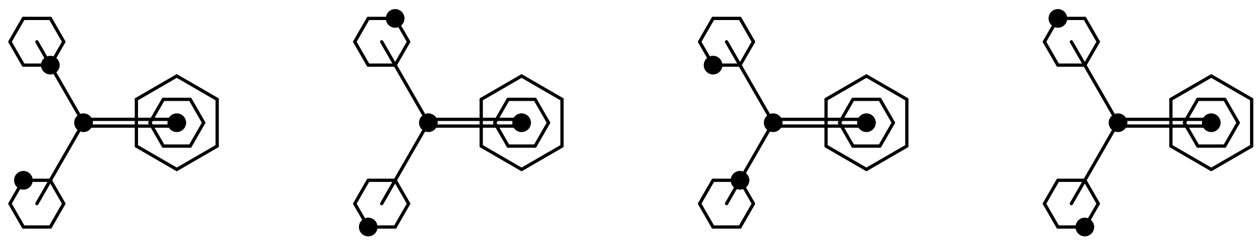
Figure 8: Interaction Diagram: $e_L^+ e_L^- \rightarrow \gamma$

The vertex diagrams can be calculated with representation theoretic tools.

- Fundamental vertex diagrams: 262144 possibilities only 8065 nonzero!
- Mathematica as a rescue: Pictorial representations
- Relative coupling constants can be calculated
- Current state of the art

Example: Electroweak interaction

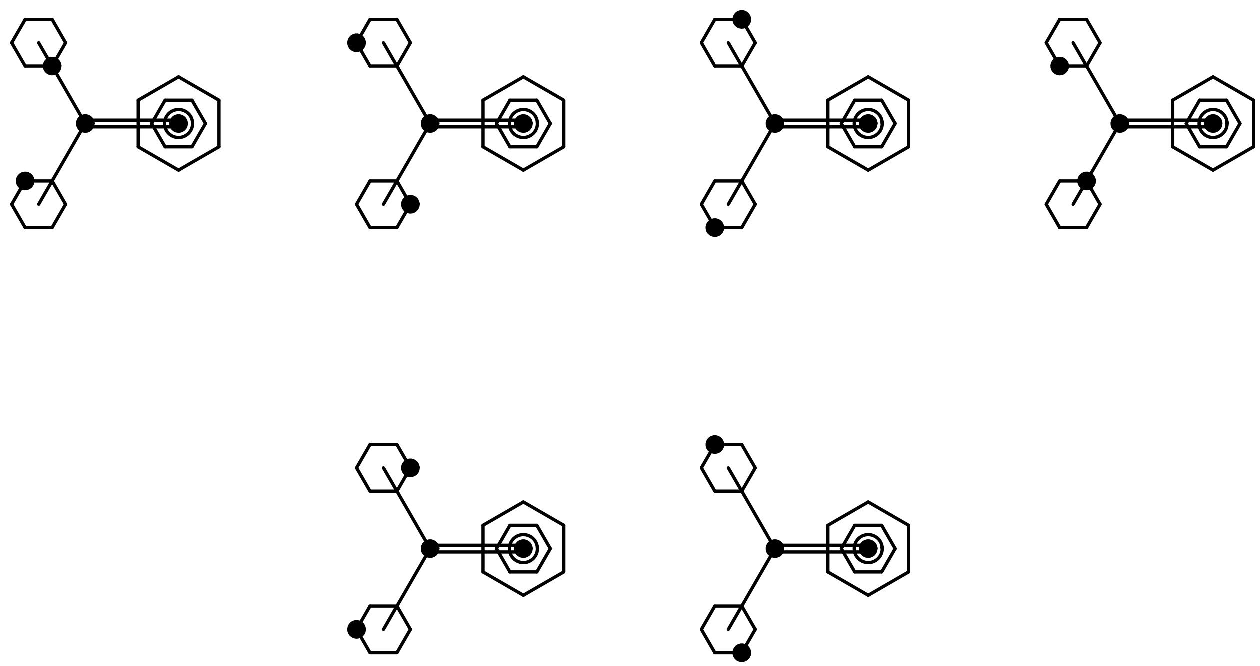
The vertex diagrams show all possible interaction diagrams of a photon and a pair of leptons.



The trace determines the coupling constant g_{ABC} of the interaction.

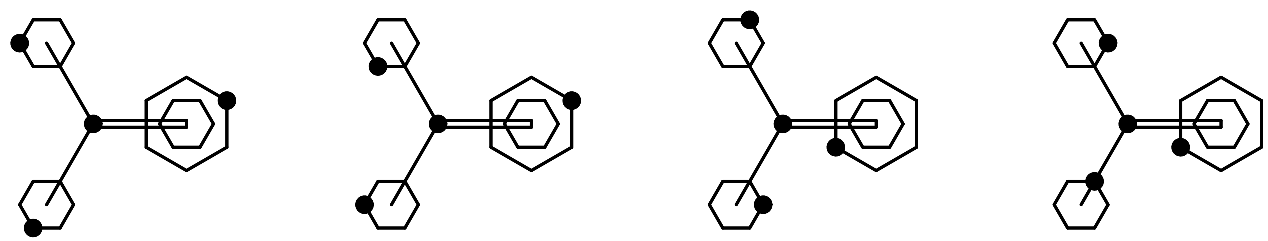
$$\mathrm{tr}(ABC) = g_{ABC}, \quad A = \gamma,\, B \in l,\, C \in l \tag{89}$$

Interaction diagrams for the neutral vector boson Z^0 with leptons.



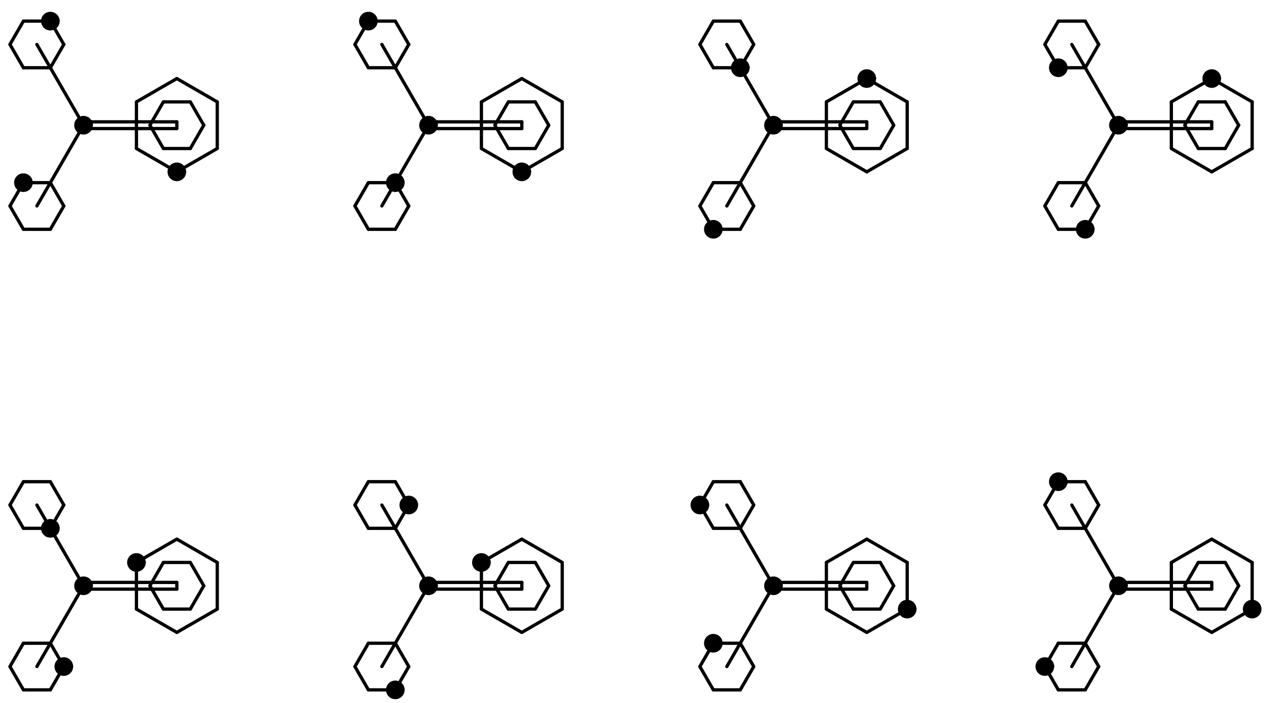
Relative coupling strength $g^2_{ABC} = 1 : 4 : 1 : 1 : 4 : 1$ for the diagramms implies a Weinberg angle $\theta_W = 30^\circ$ with $\sin^2(\theta_W) = 1/4$.

Interaction diagrams for vector bosons W^+, W^- and a pair of leptons.



The vector bosons γ, Z^0, W^+, W^- leave the decomposition of the coordinate space $(x_1, x_2, x_3) \in \mathbb{R}^3$ and the internal gauge space $(x_4, x_5, x_6, x_7) \in \mathbb{R}^4$ invariant. The remaining four vector bosons to complete the $SU(3) \supset U(1) \otimes SU(2)$ symmetry group violate this decomposition.

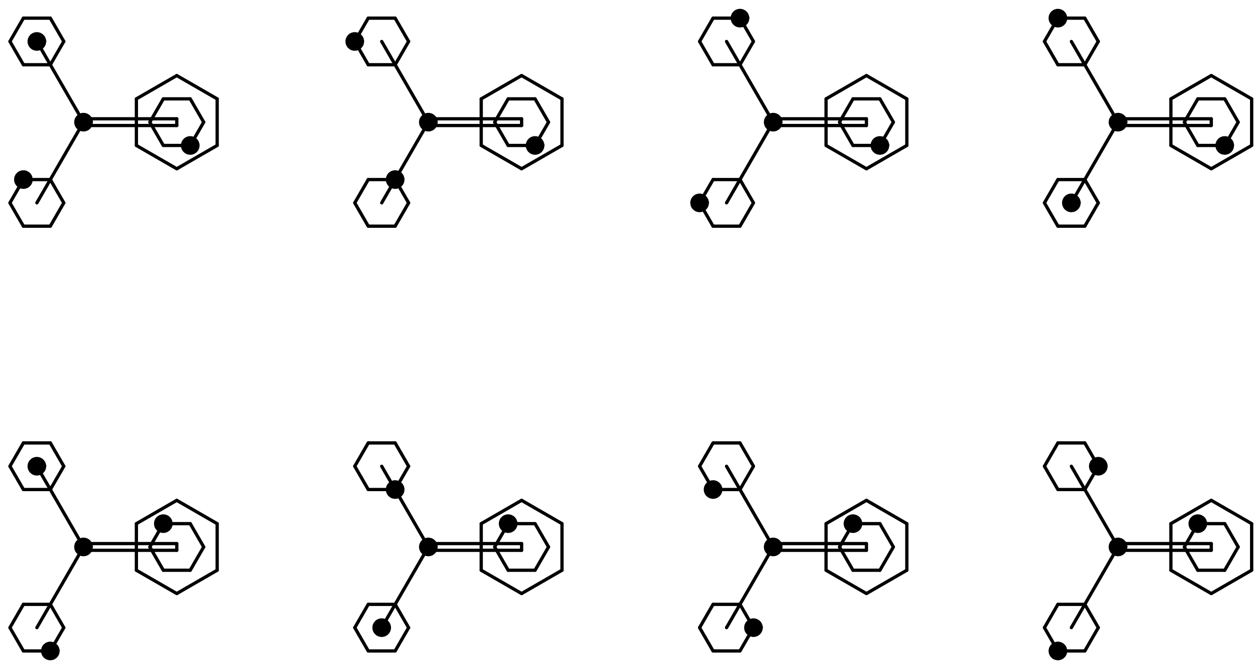
Interaction diagrams for the four vector bosons $U^{++}, U^{--}, V^{+}, V^{-}$ introduce a coupling of the right and left-handed leptons. *Qualitative similarity with the Higgs mechanism!*



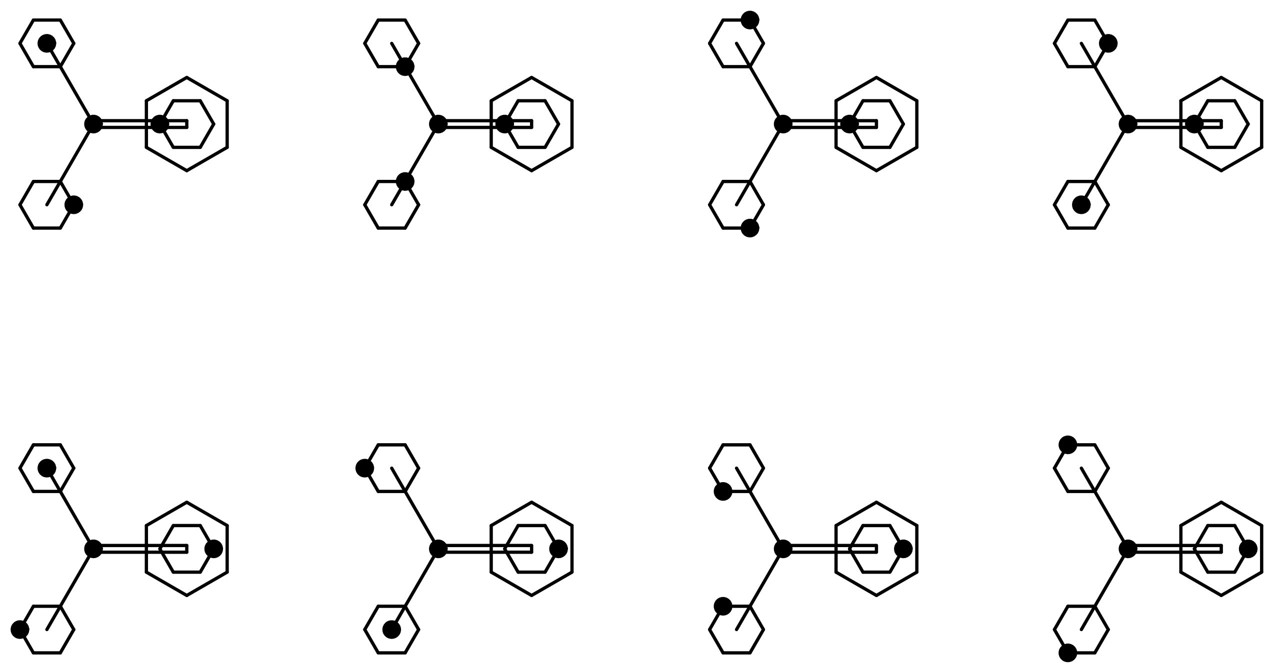
The pairs of vector bosons U, V, W are connected via a 120° rotation of the root diagram.

New symmetry principle!

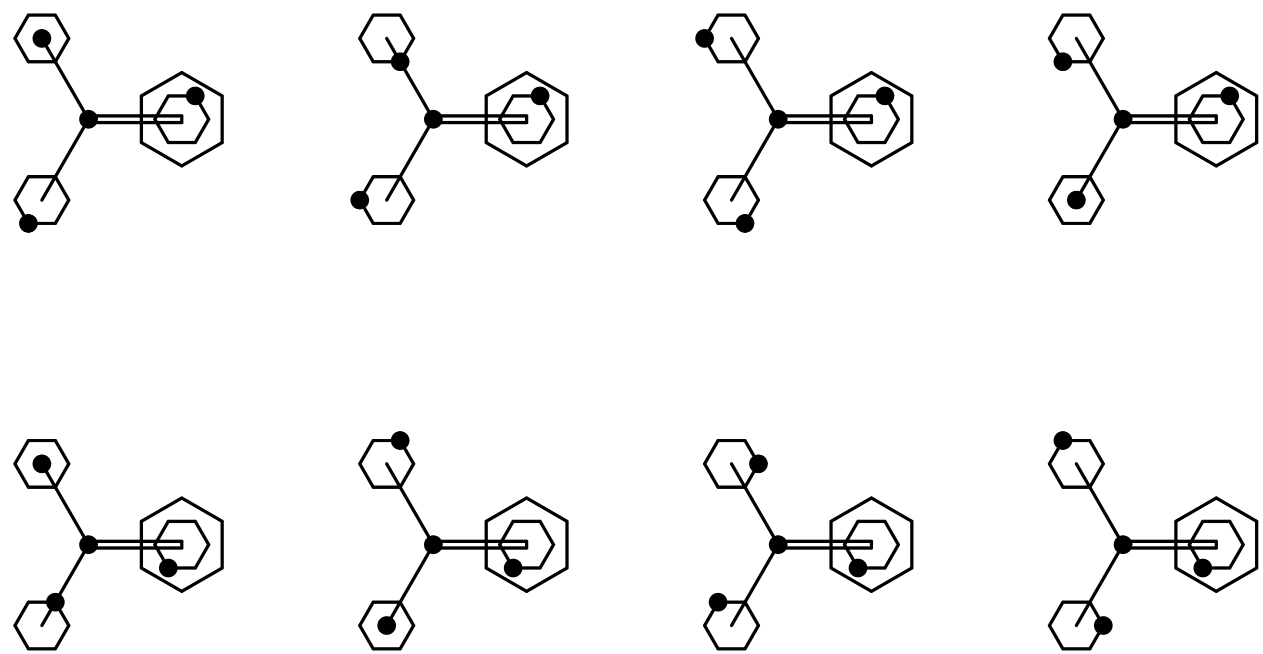
To complete the $G_2 \supset SU(3)$ symmetry the interaction diagrams of the six remaining gauge bosons are given. The first pair: $X^{\pm}$



The second pair $Y^{\pm}$



The third pair $Z^{\pm}$



These are the only bosons that couple the right handed neutrino ν_R^0.

(4) Technological Implications: Design Ideas for Interesting Devices

Use the abstract G_2 symmetry group as a design concept!

Today devices are based on electromagnetic interactions: Electrons + Photons

- Photon is only one state among 14 gauge bosons
- Use G_2 transformations to generate new blueprints for interesting devices
- Transform the matter and field content of a device under G_2

Quantum Computation: Quantum computers

Research: *Build a quantum computer based on the symmetry group* G_2*!*

- Now: Electron spin: Symmetry group $SU(2) \subset G_2$
- Use the full symmetry group G_2 acting on all leptons.
- 7 leptons + 14 gauge bosons

Quantum Computation: Breaking the quantum code

Research: *Can we break quantum cryptographic protocols?*

- Gravitons indicate the possibility to measure the angle of a gauge transformation
- Q: Does this angle information enable an attack on the protocol?
- A: Tentative yes!

Thank you!