Long-Time Limits of Quantum Dynamics

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- how does confinement of quarks in QCD come about?
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Sometimes these questions become simplified under certain additional assumptions.

An example of an additional assumption

während τ nicht merklich ändert. Zweitens ist dn proportional der Größe $f(x,t)\,dx$; dies ist ja die Zahl der Moleküle in der Volumeinheit, deren lebendige Kraft zwischen x und $x+dx$ liegt; je mehr solcher Moleküle sich in der Volumeinheit befinden, desto öfter stoßen sie in der betrachteten Weise zusammen. Drittens ist dn proportional $f(x',t)\,dx'$; denn was von dem einen der zusammenstoßenden Moleküle gilt, gilt natürlich auch vom anderen. Das Produkt dieser drei Größen muß noch multipliziert werden mit einem gewissen Proportionalitätsfaktor, von dem man leicht einsieht, daß er unendlich klein, wie $d\xi$ sein muß. Derselbe wird im allgemeinen von der Natur des Zusammenstoßes, also von den, den Zusammenstoß bestimmenden Größen x, x' und ξ abhängen. Wir wollen, um alles dies auszudrücken, den Proportionalitätsfaktor mit $d\xi \cdot \psi(x, x', \xi)$ bezeichnen, so daß wir also haben:

$$dn = \tau \cdot f(x,t)\,dx \cdot f(x',t)\,dx' \cdot d\xi\,\psi(x,x',\xi). \tag{2}$$

Dies ist das Resultat, zu dem die exakte Betrachtung des Vorganges des Zusammenstoßes führt, durch welche sich natürlich

An example of an additional assumption

... is Boltzmann's Stoßzahlansatz in his derivation of kinetic theory from mechanics (Graz, 1872).

Boltzmann could prove that the function

$$S(t) = -H(t) = -\int \mathrm{d}x \int \mathrm{d}p\, f(x,p,t) \log f(x,p,t)$$

constructed from the solution $f(x,p,t)$ of the Boltzmann equation is nondecreasing and stationary for the Maxwell–Boltzmann distribution. It is thus a natural candidate for the entropy.

$S(t)$ is the information entropy of the phase space distribution f.

Time reversal paradox (Loschmidt)

Classical mechanics is time reversal invariant ($t \to -t$)

The Boltzmann equation is not time reversal invariant

How is this possible ?

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How is this possible ?

The Stoßzahlansatz introduces this asymmetry because the momenta after the collision are correlated (Bryan, 1894).

Poincaré recurrence paradox (Zermelo)

The classical time evolution in a finite region is almost–periodic.

Thus after the Poincaré recurrence time the system comes back arbitrarily closely to its initial state.

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So why does one observe mixing of ink and water, but never the opposite phenomenon (spontaneous separation) ?

Boltzmann: because for such a large system, Poincaré's recurrence time is larger than the age of the universe.

Foundations of Statistical Mechanics

How does an irreversible transport equation emerge from classical mechanics — or quantum mechanics — and how does one get thermodynamics?

Oscar E. Lanford III proved 1976 that in a classical gas the Boltzmann equation holds for almost all initial conditions already after a small number of collisions per particle

Unfortunately, Lanford's theorem holds only for these very short times. The problem has remained open for larger times.

For quantum systems, the question of a kinetic equation is largely open, but there is a claim that near to equilibrium one can show the analogue of Lanford's theorem [Lukkarinen, Spohn, August 2008].

Results about equilibrium thermodynamics from quantum theory were obtained by Tasaki and by Fröhlich et al..

Particle in a random environment

We cannot solve the problem of the many–body time evolution

A simplified problem is the Lorentz gas: fix all but one particle at random positions and consider the motion of the one moving particle.

Quantum Lorentz model: Schrödinger's equation

$$\mathrm{i}\hbar \frac{\partial}{\partial t}\psi(t) = H\psi(t)$$

is still time–reversal invariant, because the distribution of scatterers is static.

The lattice version is the Anderson model.

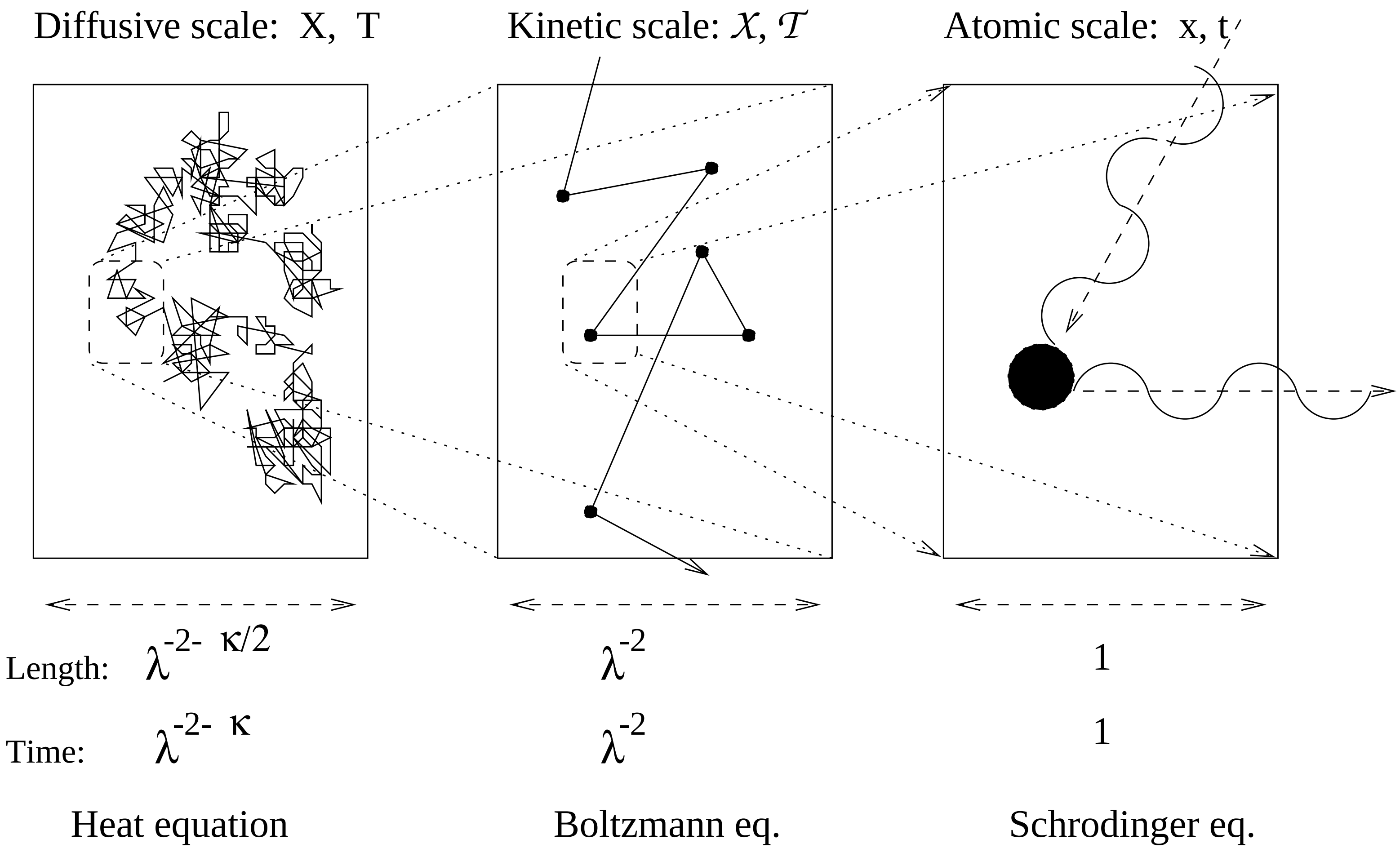
For weak scattering (small coupling)one expects diffusion, i.e. irreversible dynamics, for large times.

Diffusion in a scaling limit

Theorem [Erdös, MS, Yau]

If the coupling constant λ is small, the exact quantum mechanical time evolution of the Wigner function on time scales $t \leq \lambda^{-2-\kappa}$, $\kappa > 0$ is given, up to an error $o(\lambda)$, by the solution of a diffusion equation.

To obtain the diffusion equation, one has to follow the time evolution over $n = O(\lambda^{-\kappa})$ collisions.


Diffusive scale: X, T
Kinetic scale: $\mathcal{X}$, $\mathcal{T}$
Atomic scale: x, t
Length: $\lambda^{-2-\kappa/2}$
λ^{-2}
1
Time: $\lambda^{-2-\kappa}$
λ^{-2}
1
Heat equation
Boltzmann eq.
Schrodinger eq.

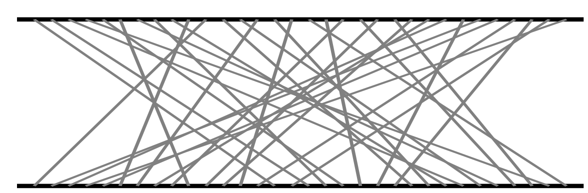
In a nutshell

The main term is given by an infinite sum of renormalized ladder graphs.

All other terms: cancellations from destructive phase interference

After n collisions, there are $O(n!)$ error terms. The destructive interference is so strong that it suppresses this $n!$.

The heart of the proof is a mapping to a problem of random permutations, and tight graph estimates in Feynman graph expansions.



Outline of the following

- Definition of the problem and physical motivation
- Main result
- Some ideas in the proof

Anderson Model

$$\mathrm{i}\frac{\partial}{\partial t}\psi(t) = H\psi(t), \quad \psi(0) = \psi_0 \qquad \text{with} \qquad H = -\frac{1}{2}\Delta + \lambda V_\omega \quad \text{on } \ell^2(\mathbb{Z}^d)$$

$-\Delta$ discrete Laplacian: $(-\Delta f)(x) = \sum_{\mu=1}^{d}(2f(x) - f(x+e_\mu) - f(x-e_\mu))$.

$$V(x) = \sum_{a\in\mathbb{Z}^d} V_a(x) \qquad V_a(x) = v_a\delta_{x,a}, \qquad v_a \text{ i.i.d. random variables.}$$

Assume that $m_k = \mathbb{E}\left(v_a^k\right)$ satisfies

$$\forall i \le 2d : m_i < \infty, \qquad m_1 = m_3 = m_5 = 0, \quad m_2 = 1.$$

In this talk, $d = 3$. Our results hold for $d \ge 3$.

Quantum Lorentz Model

$\mathrm{i}\frac{\partial}{\partial t}\psi(t) = H\psi(t), \quad \psi(0) = \psi_0$ with $H = -\frac{1}{2}\Delta + \lambda V_\omega$ on $L^2(\mathbb{R}^d)$

Δ standard Laplacian, $V_\omega(x) = \int_{\mathbb{R}^d} B(x-y)\mathrm{d}\mu_\omega(y)$

B a spherically symmetric Schwarz function with $0 \in$ supp $\hat{B}$

μ_ω a Poisson point process on $\mathbb{R}^d$ with homogeneous unit density and i.i.d. random masses.

$$\mu_\omega = \sum_{\gamma=1}^{\infty} v_\gamma(\omega)\delta_{y_\gamma(\omega)}$$

$\{y_\gamma(\omega)\}$ is Poisson, independent of the weights $\{v_\gamma(\omega)\}$

$m_k := \mathbb{E}_v\, v_\gamma^k$ satisfies

$$\forall i \leq 2d : m_i < \infty, \qquad m_1 = m_3 = m_5 = 0, \quad m_2 = 1.$$

Time evolution

Suppose the initial state is localized, i.e. $\hat{\psi}_0$ is smooth.
How does the solution $\psi(t) = \mathrm{e}^{-\mathrm{i}tH}\psi_0$ behave for large t ?

- $\lambda = 0$: $\hat{\psi}(t,k) = \mathrm{e}^{-\mathrm{i}te(k)}\hat{\psi}_0(k)$,
  with $e(k) = k^2/2$ (QLM) or $e(k) = \sum_{i=1}^{d}(1-\cos k_i)$ (AM).

$$\langle X^2\rangle_t = \langle \psi(t), X^2\psi(t)\rangle \sim t^2$$

- $\lambda \neq 0$: expect

$$\langle X^2\rangle_t = \begin{cases} O(t) & \text{diffusive} \\ O(1) & \text{localized} \end{cases}$$

depending on λ and $\hat{\psi}_0$.

Spectrum of H

- localization $\leftrightarrow$ (dense) point spectrum
- extended states $\leftrightarrow$ absolutely continuous spectrum
- $d = 1$, $\lambda > 0$: localization at all energies
  [Goldsheid, Molchanov, Pastur]
- $d \geq 2$, λ very large $\Rightarrow$ localization
  [Fröhlich–Spencer; Aizenman–Molchanov, $\ldots$]
- $d \geq 2$, λ small, but energy away from spec $-\frac{1}{2}\Delta \Rightarrow$ localization
- $d = \infty$ ($\leftrightarrow$ Cayley tree) $\Rightarrow$ extended states exist for small $\lambda > 0$.
  [Klein; Aizenman-Sims-Warzel, Froese-Hasler-Spitzer]

Major open problem

At this time there is no proof of existence of extended states in $d = 3$.

Simpler case. Randomness with a decaying envelopping function

$V_\omega(x) = \omega_x h(x)$, ω_x i.i.d., h fixed.

Theorem. [Rodnianski & Schlag; Bourgain]

$\eta > \frac{1}{2}$ and $h(x) \sim |x|^{-\eta}$ as $|x| \to \infty$

Then $H = -\Delta + V_\omega$ has absolutely continuous spectrum.

Motivations

- One–electron model of a metal with disorder
  $k \mapsto e(k)$ a band of a periodic Schrödinger operator
  V disorder
  extended vs. localized: metal–insulator transition.

- Caricature of the many–body problem
  true many–body Hamiltonian is

$$\sum_{i=1}^{n} -\frac{1}{2}\Delta_i + \lambda \sum_{i<j} v(x_i - x_j).$$

- Emergence of irreversibility from reversible dynamics

Wigner function

$$W_\psi(x,v) = \int \mathrm{d}y\ \mathrm{e}^{\mathrm{i}vy}\overline{\psi(x+\frac{y}{2})}\,\psi(x-\frac{y}{2})$$

Marginals

$$\int W_\psi(x,v)\mathrm{d}x = |\hat{\psi}(v)|^2 \qquad \int W_\psi(x,v)\mathrm{d}v = |\psi(x)|^2$$

Also, $\hat{W}_\psi(\xi,v) = \int \mathrm{d}x\ \mathrm{e}^{-\mathrm{i}x\xi}W_\psi(x,v) = \overline{\hat{\psi}(v-\xi/2)}\,\hat{\psi}(v+\xi/2)$.

$W_\psi(x,v)$ can get negative, so it is not simply a phase space density. (uncertainty principle) $\rightarrow$ Husimi function

On the lattice, one has to modify the definition of the Wigner transform slightly.

Macroscopic Scales

Ratio of typical atomic to macroscopic length scales: $\varepsilon = 10^{-8}$.

$$(\mathcal{X}, \mathcal{T}) = (\varepsilon x, \varepsilon t)$$

Velocities remain unscaled.

$$W_\psi^\varepsilon(\mathcal{X}, \mathcal{V}) = \varepsilon^{-d} W_\psi \left(\frac{\mathcal{X}}{\varepsilon}, \mathcal{V} \right)$$

The results we discuss in the following are about limits $\varepsilon \to 0$, where ε depends on λ.

Kinetic Scale

$$\eta = \lambda^2, \qquad \mathcal{T} = \eta t, \qquad \mathcal{X} = \eta x$$

Theorem. [Erdös–Yau 2000, Chen 2003]

$$\mathbb{E} W^{\eta}_{\psi(\mathcal{T}\eta^{-1})}(\mathcal{X}, \mathcal{V}) \underset{\eta \to 0}{\longrightarrow} F(\mathcal{X}, \mathcal{V}, \mathcal{T}),$$

F the solution of the *linear Boltzmann equation*

$$\frac{\partial}{\partial \mathcal{T}} F(\mathcal{X}, \mathcal{V}, \mathcal{T}) + (\nabla e)(\mathcal{V}) \cdot \nabla_{\mathcal{X}} F(\mathcal{X}, \mathcal{V}, \mathcal{T})$$
$$= 2\pi \int \mathrm{d}\mathcal{U}\; \delta(e(\mathcal{U}) - e(\mathcal{V})) \left| \hat{B}(\mathcal{U} - \mathcal{V}) \right|^2 \; [F(\mathcal{X}, \mathcal{U}, \mathcal{T}) - F(\mathcal{X}, \mathcal{V}, \mathcal{T})]$$

. .

Many–body Boltzmann equation

conjecture for the right hand side of the Boltzmann equation is, with $F_k = F(\mathcal{X}, k, \mathcal{T})$,

$$
\begin{aligned}
- \quad & 4\pi \int \mathrm{d}k_2 \, \mathrm{d}k_3 \, \mathrm{d}k_4 \, \delta(k_1 + k_2 - k_3 - k_4) \, \delta(E_1 + E_2 - E_3 - E_4) \\
& \big|\hat{v}(k_1 - k_4) - \hat{v}(k_2 - k_3)\big|^2 \\
& \Big[F_{k_1} F_{k_2} (1 - F_{k_3})(1 - F_{k_4}) - F_{k_4} F_{k_3} (1 - F_{k_2})(1 - F_{k_1})\Big]
\end{aligned}
$$

Diffusive Time Scale

$$\varepsilon = \lambda^{2+\kappa/2}, \qquad X = \varepsilon x, \qquad T = \varepsilon \lambda^{\kappa/2} t = \lambda^{\kappa+2} t$$

This is long compared to the kinetic timescale:

$$\mathcal{X} = \lambda^{-\kappa/2} X, \qquad \mathcal{T} = \lambda^{-\kappa} T$$

Theorem. [ESY]
Let $d = 3$, $\psi_0 \in \ell^2(\mathbb{Z}^3)$ and ψ_t be the solution to the random Schrödinger equation. If $\lambda > 0$ is small and if $\kappa > 0$ is small enough and $\varepsilon = \lambda^{2+\kappa/2}$, then $\mathbb{E} W^{\varepsilon}_{\psi(\lambda^{-2-\kappa}T)}$ converges weakly to the solution f of a heat equation.

More precisely: denote $\Phi(E) = \int \mathrm{d}v \, \delta(E - e(v))$ and

$$\langle F \rangle_E = \Phi(E)^{-1} \int \mathrm{d}v \; F(v) \delta(E - e(v)).$$

Main Theorem

Let $E \in [0,3]$ and $\qquad D_{ij}(E) = \frac{1}{2\pi\Phi(E)} \langle \nabla_i e \; \nabla_j e \rangle_E$

and let f be the solution of the heat equation

$$\begin{aligned} \frac{\partial}{\partial T} f(T,X,E) &= \nabla_X \cdot D(E) \nabla_X \, f(T,X,E) \\ f(0,X,E) &= \delta(X) \, \langle |\hat{\psi}_0|^2 \rangle_E \end{aligned}$$

Let $\mathcal{O}(x,v)$ be a Schwartz function on $\mathbb{R}^d \times \mathbb{T}^d$. Then

$$\begin{aligned} &\lim_{\varepsilon \to 0} \int_{(\varepsilon\mathbb{Z}/2)^d} \mathrm{d}X \int \mathrm{d}v \; \mathcal{O}(X,v) \; \mathbb{E} W^{\varepsilon}_{\psi(\lambda^{-\kappa-2}T)}(X,v) \\ &= \int_{\mathbb{R}^d} \mathrm{d}X \int \mathrm{d}v \; \mathcal{O}(X,v) \; f(T,X,e(v)). \end{aligned}$$

The limit is uniform on $[0, T_0]$ for any $T_0 > 0$.

In fact, if $\hat{\psi}_0 \in C^1$ and λ is small enough, we have

$$
\begin{aligned}
& \langle \hat{\mathcal{O}},\ \mathbb{E}\hat{W}^{\varepsilon}_{\psi(\varepsilon^{-1}\lambda^{-\kappa/2}T)} \rangle \\
= \ & \int \mathrm{d}\xi \int \Phi(E)\mathrm{d}E \ \mathrm{e}^{-(2\pi)^2 T \langle \xi,\, D(E)\xi \rangle_E} \langle \hat{\mathcal{O}}(\xi, \cdot) \rangle_E \ \langle \hat{W}_{\psi_0}(\varepsilon\xi, \cdot) \rangle_E \\
+ \ & o(\lambda)
\end{aligned}
$$

Here

$$
\langle \hat{\mathcal{O}},\ \mathbb{E}\hat{W}^{\varepsilon}_{\psi} \rangle = \int \mathrm{d}v \int \mathrm{d}\xi \ \hat{\mathcal{O}}(\xi, v) \ \mathbb{E}W^{\varepsilon}_{\psi}(\xi, v)
$$

Remarks

- The Boltzmann equation also gives the same diffusion equation in the long time limit, but it was itself derived from the QM time evolution only for shorter timescales.
- Diffusion in energy space is expected to start at $t = \lambda^{-4}$.
- $\kappa_0 = 1/6000$ for technical reasons; expected restriction of the method is $\kappa < 2$.
- Main extension of previous work is that on this time scale, the effective number of collisions per particle diverges.

Overview of the Proof

- Expansion and collision histories
- Classification of Feynman graphs
- Lowest–order renormalization
- Unitarity and expansions with remainders
- Refined classification of Feynman graphs

Expansion

$H_0 = -\frac{1}{2}\Delta \Rightarrow \psi(t) = \mathrm{e}^{-\mathrm{i}tH}\psi_0 = \sum_{n\geq 0} \psi^{(n)}(t)$,

$$\psi^{(n)}(t) \quad = \quad (-\mathrm{i}\lambda)^n \int \mathrm{d}\mu_{n+1}(s)\mathrm{e}^{-\mathrm{i}s_n H_0} V \mathrm{e}^{-\mathrm{i}s_{n-1}H_0} \ldots V \mathrm{e}^{-\mathrm{i}s_0 H_0}\psi_0$$

$$\mathrm{d}\mu_{n+1}(s) = \int\limits_{[0,\infty)^{n+1}} \mathrm{d}s_0 \ldots \mathrm{d}s_n \; \delta\left(t - \sum_{j=0}^{n} s_j\right)$$

$$V = \sum_{a\in\mathbb{Z}^d} V_a \quad \Rightarrow \quad \psi^{(n)}(t) = \sum_{\mathbf{a}_n} \psi^{(n)}_{\mathbf{a}_n}(t)$$

collision histories $\mathbf{a}_n = (a_1, \ldots, a_n) \in \mathbb{Z}^n$.

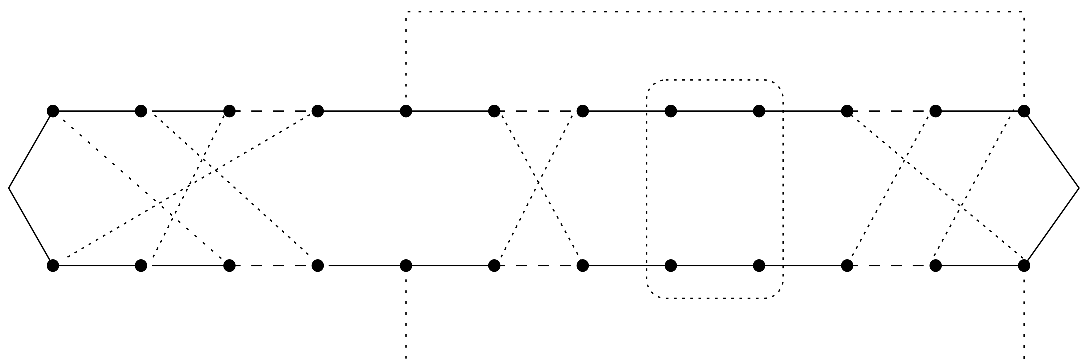
$$\hat{\psi}_n(t, p_n) = (-\mathrm{i})^n \int \prod_{j=0}^{n-1} đp_j \int \mathrm{d}\mu_{n+1}(s) \prod_{j=0}^{n} \mathrm{e}^{-\mathrm{i}s_j e(p_j)} \prod_{j=1}^{n} \hat{V}(p_j - p_{j-1})\hat{\psi}_0(p_0)$$

Disorder average and Graphs

Recall $\hat{W}_\psi(\xi, v) = \overline{\hat{\psi}(v - \xi/2)}\,\hat{\psi}(v + \xi/2)$.

$$\mathbb{E}\left[\hat{W}_{\psi(t)}(\xi, v)\right] = \sum_{n,n'} \sum_{\mathbf{a}_n, \mathbf{a}'_{n'}} \mathbb{E}\left[\hat{\psi}^{(n)}_{\mathbf{a}_n}(t, v + \xi/2)\overline{\hat{\psi}^{n'}_{\mathbf{a}'_{n'}}(t, v - \xi/2)}\right]$$

The result can be represented graphically



A graph contributing to the expansion

Particle lines get propagators $\mathrm{e}^{-\mathrm{i}s_j e(p_j)}$, interaction lines give factors λ^2.

Propagator representation

$\eta > 0$

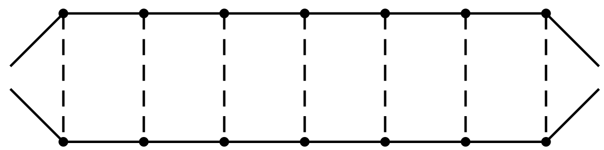
$$\begin{aligned}
&\int_{[0,\infty)^{n+1}} \mathrm{d}^{n+1}s\ \delta\left(t - \sum_{j=1}^{n+1} s_j\right) \prod_{j=0}^{n} \mathrm{e}^{-\mathrm{i}s_j e(p_j)} \\
=\quad & \mathrm{e}^{t\eta} \int_{[0,\infty)^{n+1}} \mathrm{d}^{n+1}s\ \delta\left(t - \sum_{j=0}^{n} s_j\right) \prod_{j=0}^{n} \mathrm{e}^{-\mathrm{i}s_j(e(p_j)-\mathrm{i}\eta)} \\
=\quad & \mathrm{e}^{t\eta} \int \bar{d}\alpha\ \mathrm{e}^{-\mathrm{i}t\alpha} \int_{[0,\infty)^{n+1}} \mathrm{d}^{n+1}s\ \prod_{j=0}^{n} \mathrm{e}^{-\mathrm{i}s_j(\alpha - e(p_j)+\mathrm{i}\eta)} \\
=\quad & \mathrm{i}^{-n}\mathrm{e}^{t\eta} \int \frac{\mathrm{d}\alpha}{2\pi}\ \mathrm{e}^{-\mathrm{i}\alpha t} \prod_{j=0}^{n} \frac{1}{\alpha - e(p_j) + \mathrm{i}\eta}
\end{aligned}$$

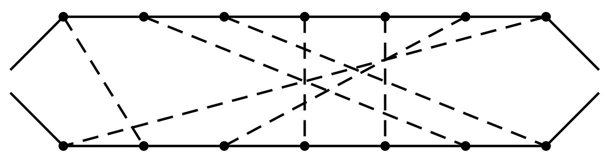
Choose $\eta = t^{-1}$.

Pairings

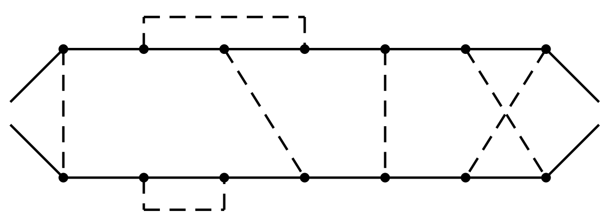
The "up–down" pairings correspond to permutations $\sigma \in \mathcal{P}_n$

$\sigma =$ id: ladder graph





Other pairings are of course possible:



Contributions at the kinetic time scale

a ladder of length n gives $\sim \frac{1}{n!}(\lambda^2 t)^n = \frac{1}{n!}\mathcal{T}^n$.

graphs with *crossings* get inverse powers of t compared to the ladder:

$$\int \mathrm{d}p \, \frac{1}{|\alpha - \omega(p) + \mathrm{i}\eta|} \, \frac{1}{|\beta - \overline{\omega(\pm p + q)} - \mathrm{i}\eta|} \leq C \frac{\eta^{-b} \, |\log \eta|^k}{\| q \| + \eta}$$

($b = 0$ for the continuum; $1/2 \leq b \leq 3/4$ on the lattice). $\| p \| = |p|$ in the continuum, $\| p \| = \min\{|p - v| : v_i \in \{0, \pm\pi\}\}$ on the lattice. Here $\omega(p) = e(p)$.

However, the number of graphs goes like $n!$, so expanding to infinite order is useful only on very short kinetic timescales

Can one do a finite order expansion? $n! t^{-1} = O(1) \Rightarrow$ roughly, $n \sim \log t$.

Duhamel formula

$$\psi(t) = \mathrm{e}^{-\mathrm{i}tH}\psi_0 = \mathrm{e}^{-\mathrm{i}tH_0}\psi_0 - \mathrm{i}\lambda \int_0^t \mathrm{d}s\, \mathrm{e}^{-\mathrm{i}(t-s)H} V \mathrm{e}^{-\mathrm{i}sH_0}\psi_0$$

Iteration gives

$$\psi(t) = \sum_{n=0}^{N-1} \psi^{(n)}(t) + \Psi_N(t),$$

$$\Psi_N(t) = (-\mathrm{i}) \int_0^t \mathrm{d}s\, \mathrm{e}^{-\mathrm{i}(t-s)H} \lambda V \psi^{(N-1)}(s)$$

$$\psi^{(n)}(t) = (-\mathrm{i}\lambda)^n \int \mathrm{d}\mu_{n+1}(s) \mathrm{e}^{-\mathrm{i}s_n H_0} V \ldots V\, \mathrm{e}^{-\mathrm{i}s_0 H_0}\psi_0$$

Unitarity of the full time evolution

$$\begin{aligned} \|\Psi_N(t)\| &\leq \int_0^t \mathrm{d}s \, \left\| \mathrm{e}^{-\mathrm{i}(t-s)H} \lambda V \psi^{(N-1)}(s) \right\| \\ &\leq \int_0^t \mathrm{d}s \, \left\| \lambda V \psi^{(N-1)}(s) \right\| \end{aligned}$$

Thus

$$\|\Psi_N(t)\|^2 \leq t \, |\lambda|^2 \int_0^t \mathrm{d}s \, \left\| V \psi^{(N-1)}(s) \right\|^2$$

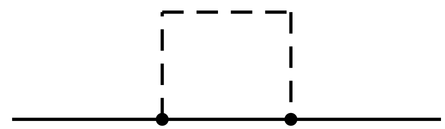
By a Schwarz inequality

$$\left| \mathbb{E}\left(\langle \hat{\mathcal{O}}, \hat{W}^{\varepsilon}_{\psi_1} \rangle - \langle \hat{\mathcal{O}}, \hat{W}^{\varepsilon}_{\psi_2} \rangle \right) \right| \leq C \int \mathrm{d}\xi \, \sup_v \left| \hat{\mathcal{O}}(\xi, v) \right| \sqrt{\mathbb{E}\|\psi_1\|^2 \, \mathbb{E}\|\psi_1 - \psi_2\|^2}$$

Thus one can get rid of H at the expense of a factor t, which can be controlled by making a further crossing explicit.

Boltzmann equation

The ladder terms give the gain term in the Boltzmann equation. The lowest order self–energy correction gives the loss term in the Boltzmann equation.
It corresponds to the “gate” graph



Long Time Scale: Renormalization

Ladders $\sim (\lambda^2 t)^n / n!$ blow up on the longer time scale. For $\varepsilon > 0$ set

$$\Theta_\varepsilon(\alpha, r) = \int \mathrm{d}q \, \frac{|\hat{B}(r-q)|^2}{\alpha - e(q) + \mathrm{i}\varepsilon},$$

$\Theta_\varepsilon(\alpha) = \Theta_\varepsilon(\alpha, r)$ for any r with $e(r) = \alpha$. Let $\Theta(\alpha) = \lim_{\varepsilon \to 0+} \Theta_\varepsilon(\alpha)$ and set

$$\theta(p) = \Theta(e(p))$$

Let $\omega(p) = e(p) + \lambda^2 \theta(p)$ and decompose

$$H = \omega(P) + U, \qquad U = \lambda V - \lambda^2 \theta(P)$$

Iterate Duhamel with $H_0 = \omega(P)$.

H_0 is not selfadjoint because ω has an imaginary part
but $\mathrm{e}^{-\mathrm{i}sH_0}$ is bounded for $s \geq 0$.

Essential features of the new propagator

For $d \geq 3$ there is $c > 0$ such that

$$\text{Im}\,\omega(p) = -\pi\Phi(e(p)) \leq -c\lambda^2 \| p \|^{d-2} \qquad .$$

There is a constant C_0 such that

$$\sup_{\alpha,\beta,r} \int \frac{\lambda^2 \; \mathrm{d}p}{(\alpha - \overline{\omega(p+r)} - i\eta)\;(\beta - \omega(p-r) + i\eta)} \; \leq 1 + C_0 \lambda^{1-O(\kappa)} \; .$$

Thus with this renormalization, the ladders become of order 1.

Expand up to order $n \sim \lambda^2 t \sim \lambda^{-\kappa}$; ladder term gives the limiting equation.

Use flexibility of the Duhamel formula to do the cancellation of the “gate” terms against the counterterm $-\lambda^2\theta(P)$ in the interaction term.

Key estimate for controlling combinatorics

For a permutation $\sigma \in \mathcal{S}_n$, let $d(\sigma)$ be the degree of the permutation, defined as the number of non–ladder indices. Let Γ_σ be the Feynman graph corresponding to σ. There is $\gamma > 0$ such that for all σ

$$|Val(\Gamma_\sigma)| \leq \lambda^{\gamma d(\sigma)}. \qquad (*)$$

The number of permutations with degree D is

$$\mathcal{N}_{n,D} = |\{\sigma \in \mathcal{S}_n : d(\sigma) = D\}| \leq 2(2n)^D$$

Expanding up to $n = O(\lambda^{-\kappa-\delta})$, $\delta > 0$, we have, if $\gamma - \kappa - \delta > 0$,

$$\sum_{\substack{\sigma \in \mathcal{S}_n \\ d(\sigma) \geq D}} \lambda^{\gamma d(\sigma)} = \sum_{d=D}^{n} \lambda^{\gamma d} \mathcal{N}_{n,d} \leq 2 \sum_{d=D}^{n} (2\lambda)^{d(\gamma-\kappa-\delta)} \leq O(\lambda^{D(\gamma-\kappa-\delta)})$$

$(*)$ is proven using a special integration algorithm for bounding the values of large Feynman graphs.

Conclusion

- Have proven diffusive behaviour on a timescale $T = \lambda^{-\kappa}\mathcal{T}$, $X = \lambda^{-\kappa/2}\mathcal{X}$.
- The number of collisions to follow in the proof is $n \sim \lambda^2 t \sim \lambda^{-\kappa}$, so effectively, need sharp bounds on Feynman graphs of arbitrary size.
- Classification in terms of degrees of permutations allows us to control combinatorics.
- Our results imply that localization is not possible in a region of size $\lambda^{-2-\delta}$, $\delta > 0$.
- Extended states conjecture remains open