Generalised parton distributions on the lattice

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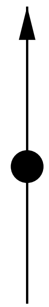
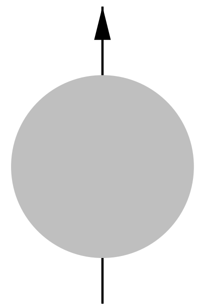
Introduction

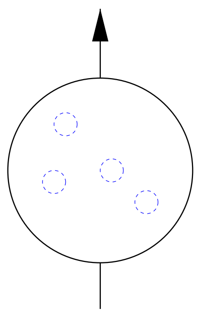
Investigating the internal structure of the nucleon



magnetic moments of proton and neutron are not those of a (structureless) Dirac particle
Nobel prize 1943 (Stern)

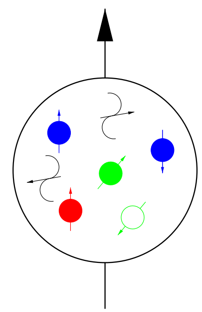


finite radius (≈ 0.86 fm), electromagnetic form factors, charge distribution
Nobel prize 1961 (Hofstadter)



deep-inelastic scattering (DIS): scaling $\rightarrow$ pointlike free constituents (partons)
Nobel prize 1990 (Friedman, Kendall, Taylor)

quark model: proton as a uud bound state



scaling violations in DIS: QCD – the (asymptotically free) quantum field theory of quarks and gluons
Nobel prize 2004 (Gross, Politzer, Wilczek)

structure of the nucleon in terms of its constituents (quarks and gluons) $\Leftrightarrow$ Quantum chromodynamics (QCD)

R.P. Feynman:

Now we were in a position that's different in history than any other time in physics, that's always different. We have a theory, a complete and definite theory of all these hadrons, and we have an enormous number of experiments and lots and lots of details, so why can't we test the theory right away to find out whether it's right or wrong? Because what we have to do is calculate the consequences of the theory. If the theory is right, what should happen, and has that happened? Well, this time the difficulty is in the first step. If the theory is right, what should happen is very hard to figure out. The mathematics needed to figure out what the consequences of this theory are have turned out to be, at the present time, insuperably difficult. At the present time–all right? And therefore it's obvious what my problem is–my problem is to try to develop a way of getting numbers out of this theory, to test it really carefully, not just qualitatively, to see if it might give the right result.

The pleasure of finding things out

QCD

quantum field theory of quarks and gluons: $\mathcal{L}_{\rm QCD} = -\frac{1}{4} F^a_{\mu\nu} F^{a,\mu\nu} + \bar{\psi} \left({\rm i} \gamma^\mu D_\mu - m \right) \psi$

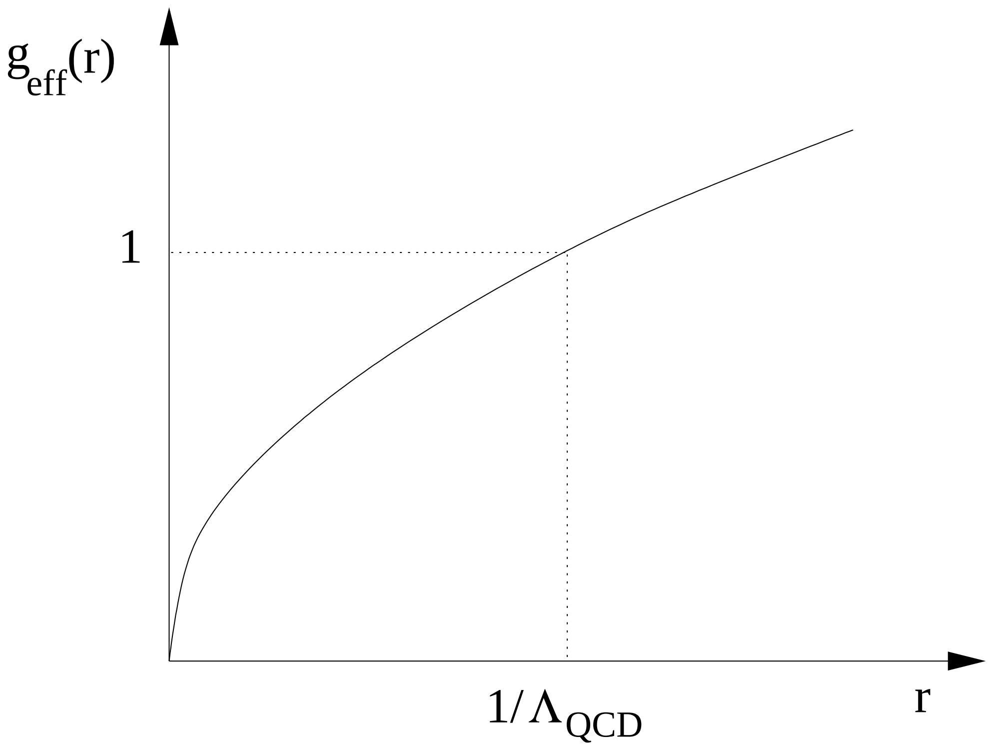
covariant derivative: $D_\mu = \partial_\mu - {\rm i} g A^a_\mu(x) T_a$

main features

- non-abelian gauge field theory with gauge group SU(3) colour
- matter fields: quarks (flavour: u, d, s, . . .) confined: no free quarks
- gauge bosons: gluons carry colour charge
- spontaneous breakdown of chiral symmetry: massive nucleon from (almost) massless quarks

small distances (high energies):
asymptotic freedom
perturbation theory

large distances (low energies):
confinement
non-perturbative methods required
nucleon as a bound state of quarks (and gluons)



Why investigate QCD at large distances?

try to answer:

- Does QCD describe the strong interaction also beyond perturbation theory?
- How does QCD manage to produce confinement?
- What is the origin of the spontaneous breakdown of chiral symmetry?
- What are the values of the hadronic matrix elements needed for the extraction of CKM matrix elements?
- What does QCD tell us about quantities which are interesting but difficult to measure experimentally? (e.g. orbital angular momentum of quarks and gluons in the nucleon)
- $\cdots$

fascinating problem:

- strongly coupled quantum field theory
- fields $\not\leftrightarrow$ physical particles

How to investigate QCD at large distances?

from first principles: Monte Carlo simulations on the lattice

starting point: path-integral formulation of quantum field theory

Minkowski space-time $\longrightarrow$ Euclidean space-time continuum $\longrightarrow$ hypercubic lattice
with lattice constant a
(non-perturbative regularisation)
classical statistical mechanics
in 4 dimensions

$$\begin{array}{ccc} \text{Minkowski space} & \underset{t=-\mathrm{i}\tau}{\longrightarrow} & \text{Euclidean space} \\ \int \mathrm{D}\Phi\, \mathrm{e}^{\mathrm{i}S[\Phi]} & \longrightarrow & \int \mathrm{D}\Phi\, \mathrm{e}^{-S_{\mathrm{E}}[\Phi]} \\ \mathrm{e}^{-\mathrm{i}Ht} & \longrightarrow & \mathrm{e}^{-H\tau} \end{array}$$

Compare with quantum mechanics

probability amplitude for the transition from y to y' in the time t:

$$\langle y'|\mathrm{e}^{-\mathrm{i}Ht}|y\rangle = \int_{\substack{x(0)=y \\ x(t)=y'}} \mathrm{D}x\, \mathrm{e}^{\mathrm{i}S[x]} = \psi(y',t)$$

($|y\rangle$ = position eigenstate)

solution of the Schrödinger equation with the initial condition $\psi(y',0) = \delta(y'-y)$

$$H = \frac{p^2}{2m} + V(x) \quad \longrightarrow \quad S[x] = \int_0^t \mathrm{d}t' \left(\frac{m}{2}\dot{x}(t')^2 - V(x(t'))\right)$$

Hamilton operator — classical action

to get rid of the oscillating integrand: continue analytically to Euclidean times $t = -\mathrm{i}\tau$, $\tau > 0$

$$\langle y'|\mathrm{e}^{-H\tau}|y\rangle = \int_{\substack{x(0)=y \\ x(\tau)=y'}} \mathrm{D}x\, \mathrm{e}^{-S_{\mathrm{E}}[x]}$$

with $$S_{\mathrm{E}}[x] = \int_0^\tau \mathrm{d}\tau' \left(\frac{m}{2}\dot{x}(\tau')^2 + V(x(\tau'))\right)$$ (Euclidean action) partition function (statistical mechanics!) as a path integral

$$Z(\tau) = \mathsf{Tr}\left(\mathrm{e}^{-H\tau}\right) = \int \mathrm{d}y\, \langle y|\mathrm{e}^{-H\tau}|y\rangle = \int \mathrm{d}y \int_{\substack{x(0)=y \\ x(\tau)=y}} \mathrm{D}x\, \mathrm{e}^{-S_\mathrm{E}[x]} = \int_{x(0)=x(\tau)} \mathrm{D}x\, \mathrm{e}^{-S_\mathrm{E}[x]}$$

or evaluated with the help of a complete set of energy eigenstates
$H|n\rangle = |n\rangle$, $E_0 < E_1 < E_2 < \cdots$

$$Z(\tau) = \mathsf{Tr}\left(\mathrm{e}^{-H\tau}\right) = \sum_{n=0}^{\infty} \langle n|\mathrm{e}^{-H\tau}|n\rangle = \sum_{n=0}^{\infty} \mathrm{e}^{-E_n\tau} \underset{\tau\to\infty}{\longrightarrow} \mathrm{e}^{-E_0\tau}$$

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analogously for $\tau > \tau_1 > \tau_2 > 0$

$$\lim_{\tau\to\infty} \frac{1}{Z(\tau)} \mathsf{Tr}\left(\mathrm{e}^{-H(\tau-\tau_1+\tau_2)} x \mathrm{e}^{-H(\tau_1-\tau_2)} x\right) = \sum_{n=0}^{\infty} |\langle 0|x|n\rangle|^2 \mathrm{e}^{-(E_n-E_0)(\tau_1-\tau_2)}$$

$$= \lim_{\tau\to\infty} \frac{\int_{x(0)=x(\tau)} \mathrm{D}x\, \mathrm{e}^{-S_\mathrm{E}[x]} x(\tau_1) x(\tau_2)}{\int_{x(0)=x(\tau)} \mathrm{D}x\, \mathrm{e}^{-S_\mathrm{E}[x]}} = \lim_{\tau\to\infty} \langle x(\tau_1) x(\tau_2)\rangle$$

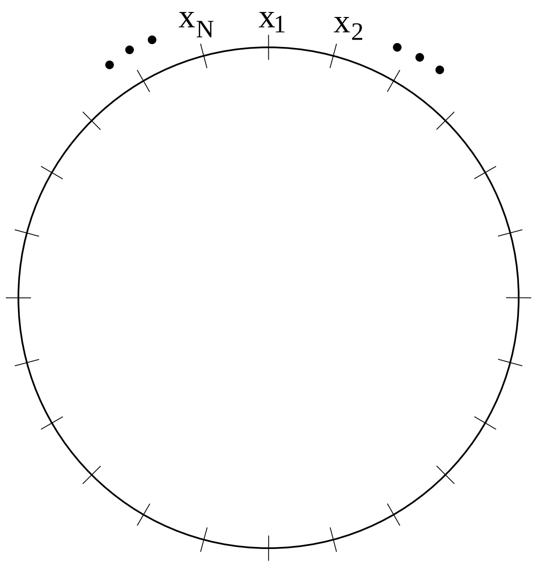
correlation function

$\longrightarrow$ decay of Euclidean correlation functions contains information about the energy spectrum and matrix elements to be computed: $$\frac{\int_{x(0)=x(\tau)} \mathrm{D}x \, \mathrm{e}^{-S_{\mathrm{E}}[x]} x(\tau_1) x(\tau_2)}{\int_{x(0)=x(\tau)} \mathrm{D}x \, \mathrm{e}^{-S_{\mathrm{E}}[x]}}$$

similarity with classical statistical mechanics!

numerical evaluation of the functional integrals by Monte Carlo simulation of a lattice approximation:

$$\int_{x(0)=x(\tau)} \mathrm{D}x \, \mathrm{e}^{-S_{\mathrm{E}}[x]} A[x] \rightarrow \int \mathrm{d}x_1 \cdots \mathrm{d}x_N \, \mathrm{e}^{-S_{\mathrm{E}}^{\mathrm{L}}(x_1,\ldots,x_N)} A^{\mathrm{L}}(x_1,\ldots,x_N)$$



circumference τ, $\Delta\tau = \tau/N$

e.g. (anti-)periodic boundary conditions

Markov chain of configurations $(x_1,\ldots,x_N)^{(1)} \rightarrow (x_1,\ldots,x_N)^{(2)} \rightarrow (x_1,\ldots,x_N)^{(3)} \rightarrow \cdots$

asymptotic distribution $\propto \mathrm{e}^{-S_{\mathrm{E}}^{\mathrm{G}}(x_1,\ldots,x_N)}$

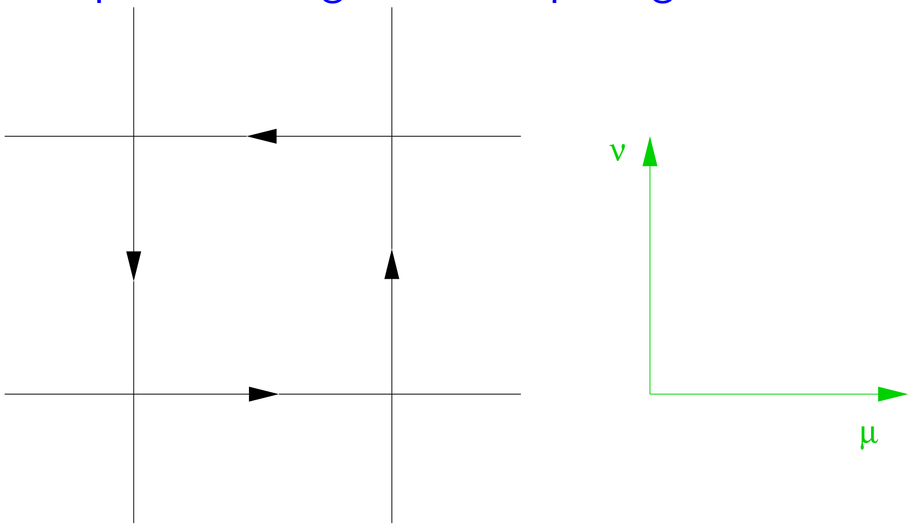
Back to QCD

discretisation of Euclidean QCD: keep gauge invariance exact!

gauge fields (gluons): parallel transporters $U(x,\mu) \in$ SU(3) along the link $x \longrightarrow x + \hat{\mu}$

gauge transformations on the sites: $U(x,\mu) \longrightarrow g(x)U(x,\mu)g^{\dagger}(x+\hat{\mu})$

action in terms of parallel transporters along closed loops, e.g.



$$S_{\mathrm{gauge}} = -\frac{1}{g^2}\sum_{x,\mu,\nu} \mathrm{tr}\left(U(x,\mu)U(x+\hat{\mu},\nu)U^{+}(x+\hat{\nu},\mu)U^{+}(x,\nu)\right)$$

$\hat{\mu}$: vector of length a in direction μ, (bare) gauge coupling $g \longrightarrow \beta = 6/g^2$

fermion fields (quarks):

(Grassmann-valued) Dirac spinors ψ, $\bar\psi$ on the lattice sites

action for Wilson fermions:

$$S_{\rm Wilson} = a^4 \sum_x \left\{ \frac{1}{2a} \sum_\mu \bar\psi(x)\gamma_\mu \left[U(x,\mu)\psi(x+\hat\mu) - U^\dagger(x-\hat\mu,\mu)\psi(x-\hat\mu) \right] + m\bar\psi(x)\psi(x) + \frac{r}{2a} \sum_\mu \bar\psi(x) \left[2\psi(x) - U(x,\mu)\psi(x+\hat\mu) - U^\dagger(x-\hat\mu,\mu)\psi(x-\hat\mu) \right] \right\}$$

$r = 1$ in the numerical results

abbreviation: $S_{\rm Wilson} = \bar\psi M[U]\psi$ with the (huge) fermion matrix M

(carrying position (x), spinor and colour indices)

in our simulations: $O(a)$-improved Wilson fermions = clover fermions expectation value of the observable A:

$$\langle A \rangle = Z^{-1} \int \mathrm{D}U \, \mathrm{D}\bar{\psi} \, \mathrm{D}\psi \; A(U, \bar{\psi}, \psi) \mathrm{e}^{-S_{\mathrm{gauge}}[U] - \bar{\psi} M[U] \psi}$$

with the partition function $Z = \int \mathrm{D}U \, \mathrm{D}\bar{\psi} \, \mathrm{D}\psi \, \mathrm{e}^{-S_{\mathrm{gauge}}[U] - \bar{\psi} M[U] \psi}$

integration over the (Grassmann-valued) fermion fields $\longrightarrow \det M$, M^{-1}

$$Z = \int \mathrm{D}U \; \underbrace{\det M[U] \, \mathrm{e}^{-S_{\mathrm{gauge}}[U]}}_{\mathrm{e}^{-S_{\mathrm{eff}}[U]}}$$

$$\langle A \rangle = Z^{-1} \int \mathrm{D}U \, (M[U]^{-1})_{ab} (M[U]^{-1})_{cd} \cdots \, \mathrm{e}^{-S_{\mathrm{eff}}[U]}$$

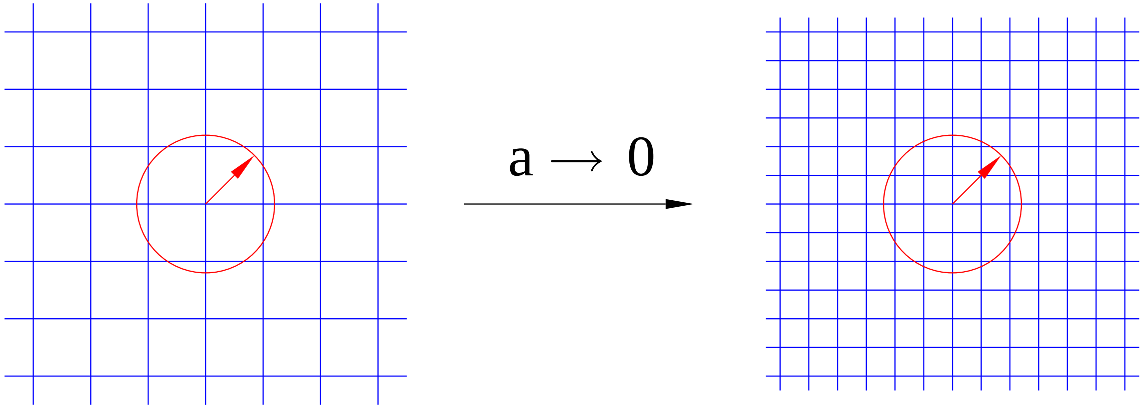
looks like classical statistical mechanics in 4 dimensions!

compute $\langle A \rangle = Z^{-1} \int \mathrm{D}U \; A(U) \mathrm{e}^{-S_{\mathrm{eff}}[U]}$ by Monte Carlo methods

computable: energies (masses), matrix elements of (composite) operators such as $\bar{\psi}(x)\psi(x)$

Continuum limit

a dependence of the bare parameters (e.g. g) such that the physical observables are finite as $a \longrightarrow 0$



keep suitable quantity fixed in physical units, e.g., $r_0 \approx 0.5\,\text{fm}$

$$R^2 \frac{\mathrm{d}V(R)}{\mathrm{d}R}\bigg|_{R=r_0} = 1.65 \qquad V(R) = \text{ heavy-quark potential}$$

asymptotic freedom: $a \longrightarrow 0 \Leftrightarrow g \longrightarrow 0$

additionally: (multiplicative) renormalisation of composite operators

How to describe the internal structure of the nucleon?

hydrogen atom: proton + electron	proton: 3 valence quarks + 1, 2, . . . quark-antiquark pairs + 1, 2, . . . gluons
nonrelativistic quantum mechanics	quantum field theory

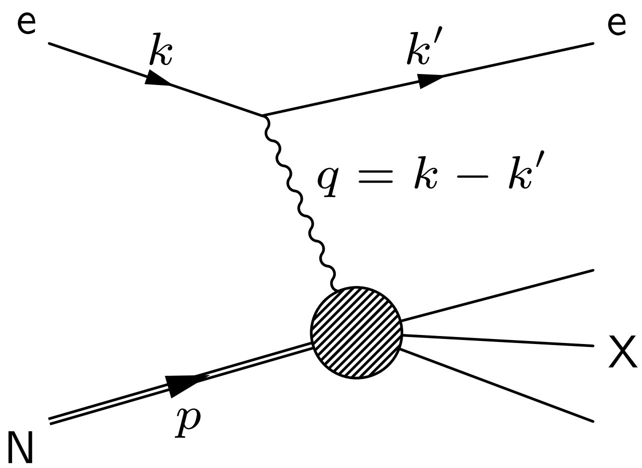
information on the internal structure of the nucleon from, e.g., lepton-nucleon scattering experiments

exclusive: electromagnetic form factors (FFs) ↘

generalised parton distributions (GPDs)

inclusive: structure functions, parton densities ↗

deep-inelastic scattering (DIS):



kinematic variables: $Q^2 = -q^2$, $x = Q^2/(2p \cdot q)$ $(0 \leq x \leq 1)$

unpolarised structure functions	$F_1(x, Q^2)\ ,\ F_2(x, Q^2)$	from inclusive cross section
polarised structure functions	$g_1(x, Q^2)\ ,\ g_2(x, Q^2)$	

deep-inelastic limit: $Q^2 \to \infty$, x fixed $\Rightarrow$ scaling consider moments in the deep-inelastic limit:

$$2\int_0^1 dx\, x^{n-1} F_1(x,Q^2) = \sum_{q=u,d} c_{1,n}^{(q)}(Q^2/\mu^2, g(\mu))\, v_n^{(q)}(\mu) + \cdots$$

for the proton $(n = 2, 4, \ldots)$ F_2 similar

Wilson coefficients $c_{1,n}^{(q)}$ calculated in perturbation theory

proton (forward) matrix elements $v_n^{(q)}(\mu)$ (μ: renormalisation scale)

$$\frac{1}{2}\sum_s \langle p, s|\mathcal{O}^q_{(\mu_1\cdots\mu_n)}|p, s\rangle = 2v_n^{(q)} p_{(\mu_1}\cdots p_{\mu_n)}$$

$$\mathcal{O}^q_{\mu_1\cdots\mu_n} = (\mathrm{i}/2)^{n-1}\; \bar{q}\gamma_{\mu_1}\overleftrightarrow{D}_{\mu_2}\cdots\overleftrightarrow{D}_{\mu_n} q \quad \text{with} \quad \overleftrightarrow{D}_\mu = \overrightarrow{D}_\mu - \overleftarrow{D}_\mu$$

$(\cdots)$: symmetrisation and subtraction of trace terms D_μ: covariant derivative

$\longrightarrow$ twist-2 operators (dominating in the deep-inelastic limit)

parton model: interpretation in terms of parton densities (parton distributions) $q(x)$, $\bar{q}(x)$

"probability" to find a quark (antiquark) with momentum fraction x

$$v_n^{(q)} = \langle x^{n-1}\rangle^{(q)} = \int_0^1 \mathrm{d}x\, x^{n-1}\left(q(x) + (-1)^n \bar{q}(x)\right)$$

similarly in the polarised case:

$$\int_0^1 dx\, x^n g_1(x, Q^2) = \frac{1}{4} \sum_{q=u,d} e_{1,n}^{(q)}(Q^2/\mu^2, g(\mu)) a_n^{(q)}(\mu) + \cdots$$

$(n = 0, 2, 4, \ldots)$

operators: $\mathcal{O}^{q,5}_{\sigma\mu_1\cdots\mu_n} = (\mathrm{i}/2)^n \; \bar{q}\gamma_\sigma \gamma_5 \overleftrightarrow{D}_{\mu_1} \cdots \overleftrightarrow{D}_{\mu_n} q$

$$\langle p, s | \mathcal{O}^{q,5}_{(\sigma\mu_1\cdots\mu_n)} | p, s \rangle = a_n^{(q)} s_{(\sigma} p_{\mu_1} \cdots p_{\mu_n)}$$

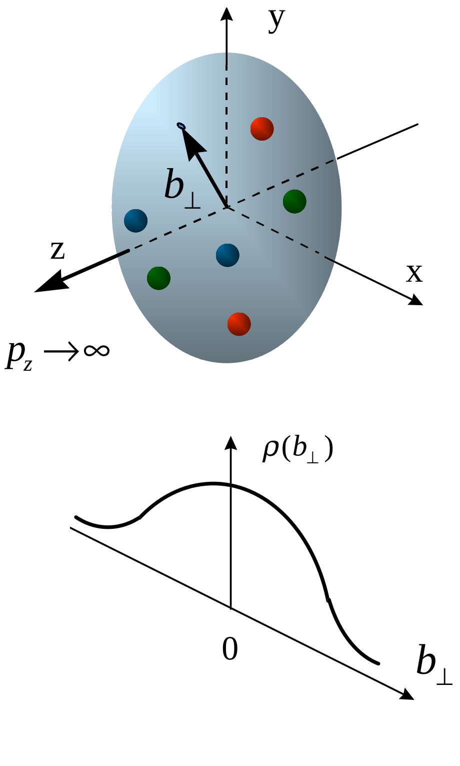
parton interpretation of a_n:

$$a_n^{(q)} = 2 \int_0^1 \mathrm{d}x\, x^n \Big[\underbrace{q_+(x) - q_-(x)}_{\Delta q(x)} + (-1)^n (\bar{q}_+(x) - \bar{q}_-(x)) \Big]$$

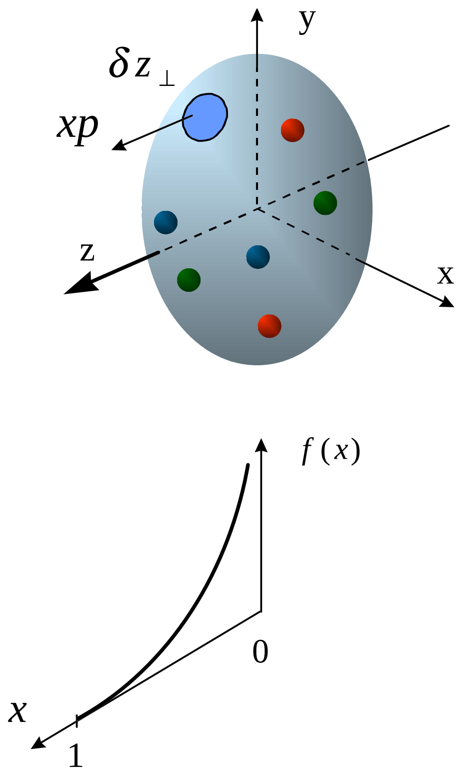
$q_+(x)$ ($q_-(x)$): "probability" to find a quark with momentum fraction x and helicity equal (opposite) to that of the nucleon

generalisation: generalised parton distributions (GPDs)

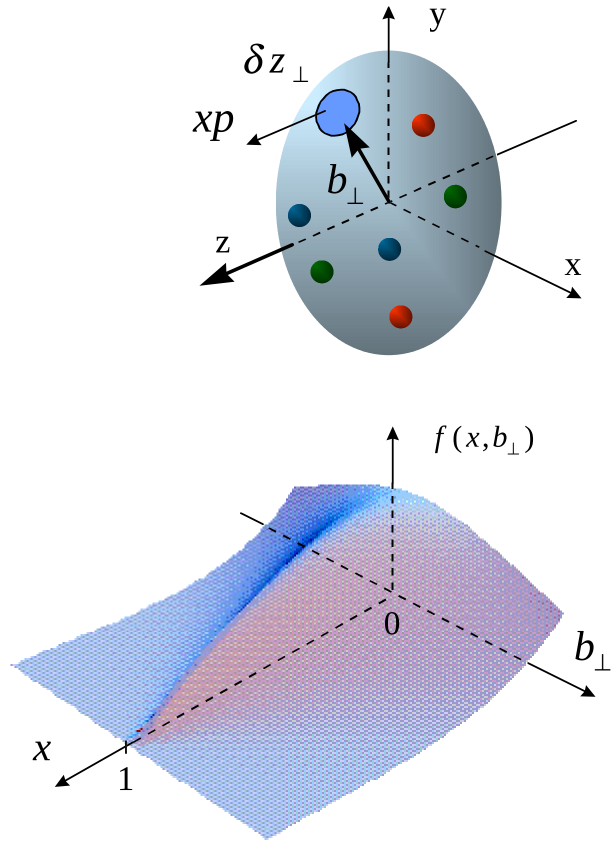
form factor



parton density



GPD at $\xi = 0$

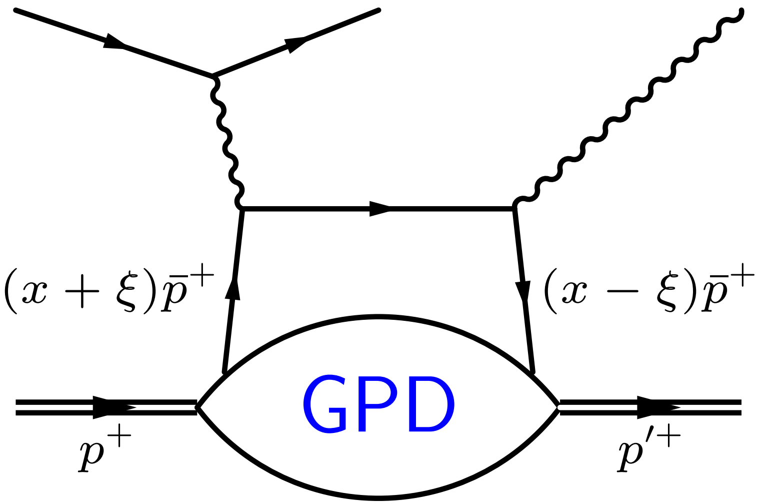


pictures by Dieter Müller

probabilistic interpretation in impact parameter space (M. Burkardt)

expect: $f(x, \vec{b}_\perp) \stackrel{x \to 1}{\longrightarrow} \delta(\vec{b}_\perp)$

Formal definition of GPDs



Wilson line
↓

$$\int \frac{\mathrm{d}\lambda}{2\pi} \mathrm{e}^{\mathrm{i}\lambda x} \langle p' | \bar{q}(-\tfrac{1}{2}\lambda n) \not{n}\, \mathcal{U} q(\tfrac{1}{2}\lambda n) | p \rangle$$

$$= H_q(x,\xi,t)\bar{u}(p')\not{n} u(p)$$

$$+ E_q(x,\xi,t)\bar{u}(p')\frac{\mathrm{i}\sigma^{\mu\nu} n_\mu \Delta_\nu}{2M_N} u(p)$$

dependence on renormalisation scale suppressed

$\bar{p} = \frac{1}{2}(p' + p)$, $\Delta = p' - p$, n: light-like vector with $\bar{p} \cdot n = 1$

- nonlocal (along the light cone) operator: difficult (impossible) on the lattice
  $\longrightarrow$ consider moments w.r.t. x
- matrix element with nonzero momentum transfer Δ
- GPDs depend on the three variables x, $\xi = -n \cdot \Delta/2$, $t = \Delta^2$ (and the renormalisation scale)

normal parton distributions: $\Delta = 0$ (forward matrix elements) $\Rightarrow \xi = 0$

limiting cases: $$H_q(x,0,0) = \begin{cases} q(x) & \text{for } x > 0 \\ -\bar{q}(-x) & \text{for } x < 0 \end{cases}$$

$$\int_{-1}^{1} \mathrm{d}x\, H_q(x,\xi,t) = F_1^q(t) \qquad \int_{-1}^{1} \mathrm{d}x\, E_q(x,\xi,t) = F_2^q(t)$$

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probabilistic interpretation in impact parameter space (M. Burkardt)

$\xi = 0$: momentum transfer purely transverse $\Delta = \Delta_\perp$

$$q(x,\vec{b}_\perp) = \int \frac{\mathrm{d}^2\Delta_\perp}{(2\pi)^2}\, \mathrm{e}^{\mathrm{i}\vec{b}_\perp\cdot\vec{\Delta}_\perp} H_q(x,0,-\vec{\Delta}_\perp^2)$$

with $\int \mathrm{d}^2 b_\perp q(x,\vec{b}_\perp) = q(x)$ expect: $q(x,\vec{b}_\perp) \stackrel{x\to 1}{\longrightarrow} \delta(\vec{b}_\perp)$

Note: momentum fraction of the quarks fixed

$\longrightarrow$ longitudinal position undetermined (Heisenberg)

$\longrightarrow$ distribution in impact parameter space meaningful for a lattice calculation:

need expression in terms of matrix elements of local operators

$\longrightarrow$ consider moments with respect to x

$$\int_{-1}^{1} \mathrm{d}x\, x^{n-1} H_q(x,\xi,t) = \sum_{i=0}^{\left[\frac{n-1}{2}\right]} A^q_{n,2i}(t)(-2\xi)^{2i} + \mathrm{Mod}(n+1,2) C^q_n(t)(-2\xi)^n$$

$$\int_{-1}^{1} \mathrm{d}x\, x^{n-1} E_q(x,\xi,t) = \sum_{i=0}^{\left[\frac{n-1}{2}\right]} B^q_{n,2i}(t)(-2\xi)^{2i} - \mathrm{Mod}(n+1,2) C^q_n(t)(-2\xi)^n$$

A, B, C: generalised form factors (GFFs)

special cases: $A^q_{1,0}(t) = F^q_1(t) \quad , \quad B^q_{1,0}(t) = F^q_2(t)$

contribution of flavour q to the electromagnetic form factors

$$\int_{-1}^{1} \mathrm{d}x\, x^{n-1} H_q(x,\xi=0,t) = A^q_{n0}(t)$$

matrix elements of local operators

$$\langle p'|\mathcal{O}^q_{\mu_1\cdots\mu_n}|p\rangle = \bar{u}(p')\gamma_{(\mu_1}u(p)\sum_{i=0}^{\left[\frac{n-1}{2}\right]} A^q_{n,2i}(t)\Delta_{\mu_2}\cdots\Delta_{\mu_{2i+1}}\overline{p}_{\mu_{2i+2}}\cdots\overline{p}_{\mu_n)}$$
$$-\frac{\bar{u}(p')\mathrm{i}\Delta^\alpha\sigma_{\alpha(\mu_1}u(p)}{2M_N}\sum_{i=0}^{\left[\frac{n-1}{2}\right]} B^q_{n,2i}(t)\Delta_{\mu_2}\cdots\Delta_{\mu_{2i+1}}\overline{p}_{\mu_{2i+2}}\cdots\overline{p}_{\mu_n)}$$
$$+\,C^q_n(t)\mathrm{Mod}(n+1,2)\frac{1}{M_N}\bar{u}(p')u(p)\Delta_{(\mu_1}\cdots\Delta_{\mu_n)}$$

$(\cdots)$: symmetrisation and subtraction of trace terms

twist-2 operators: $\mathcal{O}^q_{\mu_1\cdots\mu_n} = \left(\frac{\mathrm{i}}{2}\right)^{n-1}\bar{q}\gamma_{(\mu_1}\overleftrightarrow{D}_{\mu_2}\cdots\overleftrightarrow{D}_{\mu_n)}q$ with $\overleftrightarrow{D} = \overrightarrow{D} - \overleftarrow{D}$

analogous equations in the polarised case: $\mathcal{O}^{q,5}_{\mu_1\cdots\mu_n} = (\mathrm{i}/2)^{n-1}\,\bar{q}\gamma_{\mu_1}\gamma_5\overleftrightarrow{D}_{\mu_2}\cdots\overleftrightarrow{D}_{\mu_n}q$

with form factors $\tilde{A}$, $\tilde{B}$, $\tilde{C}$ and corresponding GPDs

and for $(\mathrm{i}/2)^n\,\bar{q}\mathrm{i}\sigma_{\mu\nu}\overleftrightarrow{D}_{\mu_1}\cdots\overleftrightarrow{D}_{\mu_n}q$

similar towers of gluon operators

particularly convenient for a lattice calculation: flavour non-singlet (isovector) operators $(u-d)$

(quark-line) disconnected contributions and gluonic operators drop out

Calculating nucleon matrix elements on the lattice

interpolating field for the proton (neutron by interchanging u and d)

$$B_\alpha(t,\vec{p}) = \sum_{x,\, x_4=t} \mathrm{e}^{-\mathrm{i}\vec{p}\cdot\vec{x}} \epsilon_{ijk} u^i_\alpha(x) u^j_\beta(x) (C^{-1}\gamma_5)_{\beta\gamma} d^k_\gamma(x)$$

$$\bar{B}_\alpha(t,\vec{p}) = \sum_{x,\, x_4=t} \mathrm{e}^{\mathrm{i}\vec{p}\cdot\vec{x}} \epsilon_{ijk} \bar{d}^i_\beta(x) (\gamma_5 C)_{\beta\gamma} \bar{u}^j_\gamma(x) \bar{u}^k_\alpha(x)$$

$+$ smearing

$$C\gamma_\mu^{\mathrm{T}} C^{-1} = -\gamma_\mu$$

On a lattice with time extent T we have for the nucleon 2-point function:

$$\langle B(t)\bar{B}(0)\rangle \stackrel{T\to\infty}{=} \langle 0|B\mathrm{e}^{-Ht}\bar{B}|0\rangle \stackrel{t\to\infty}{=} \langle 0|B|N\rangle \mathrm{e}^{-E_N t}\langle N|\bar{B}|0\rangle + \cdots$$

momentum 0: $E_N = M_N$

3-point function with $t > \tau > 0$:

$$\begin{aligned}\langle B(t)\mathcal{O}(\tau)\bar{B}(0)\rangle &\stackrel{T\to\infty}{=} \langle 0|B\mathrm{e}^{-H(t-\tau)}\mathcal{O}\mathrm{e}^{-H\tau}\bar{B}|0\rangle \\ &= \langle 0|B|N\rangle \mathrm{e}^{-E_N(t-\tau)}\langle N|\mathcal{O}|N\rangle \mathrm{e}^{-E_N\tau}\langle N|\bar{B}|0\rangle + \cdots \\ &= \langle 0|B|N\rangle \mathrm{e}^{-E_N t}\langle N|\bar{B}|0\rangle \langle N|\mathcal{O}|N\rangle + \cdots\end{aligned}$$

ratio (for forward matrix elements) $R \equiv \dfrac{\langle B(t)\mathcal{O}(\tau)\bar{B}(0)\rangle}{\langle B(t)\bar{B}(0)\rangle} = \langle N|\mathcal{O}|N\rangle + \cdots$

(with an appropriate "projection" on the spinor indices)

should be independent of τ, t for $0 \ll \tau \ll t$ i.e. if excited states can be neglected.

after these long preparations:

write simulation codes and run them on a sufficiently powerful computer . . .

Hitachi SR8000 (LRZ München)
used by QCDSF (among others)

. . . and wait what happens . . .

Ein Hase im Rechenzentrum

Die schnellste Maschine,
Parallelarchitektur,
knapp tausend Megaflops,
vermag seinem kleinen Gehirn
nicht zu folgen.

Die bebende Oberlippe
zuckend im Neonlicht,
die großen Augen starr
auf den Bildschirm gerichtet,
trommelt er panisch
gegen das graue Linoleum.

Dann, es ist drei Uhr früh,
der letzte Plasmaphysiker
ist nach Hause gegangen,
schnellt er plötzlich hoch
und jagt im Zickzack
zwischen Monitoren
und stotternden Druckern
durch den verlassenen Raum.

Weicher Feigling,
fünfzig Millionen Jahre
älter als wir!
Dem Blutdurst der Jäger,
der Ramme, dem Gas,
dem Virus entkommen,
schlägt er ungerührt seine Haken.

Aus dem Eozän hoppelt er
an uns vorbei in eine Zukunft,
reich an Feinden,
doch nahrhaft und geil
wie der Löwenzahn.

Hans Magnus Enzensberger
Gedichte 1950 – 2000

The simulations

non-perturbatively $O(a)$-improved Wilson (clover) fermions
scale set by $r_0 = 0.467$ fm (after chiral extrapolation of r_0/a for each β)

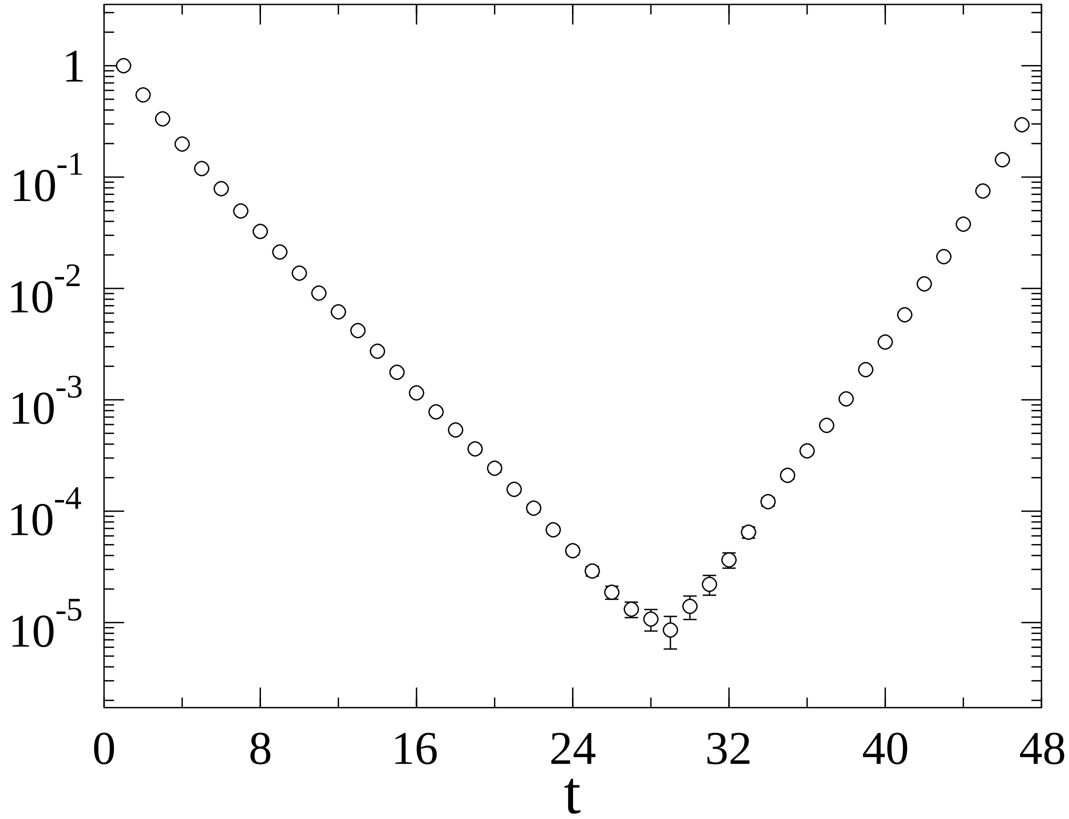
β	κ	Size	a [fm]	L [fm]	m_π [GeV]
5.20	0.13420	$16^3 \times 32$	0.086	1.37	1.345(18)
5.20	0.13500	$16^3 \times 32$	0.086	1.37	0.954(13)
5.20	0.13550	$16^3 \times 32$	0.086	1.37	0.669(9)
5.25	0.13460	$16^3 \times 32$	0.080	1.28	1.219(18)
5.25	0.13520	$16^3 \times 32$	0.080	1.28	0.945(14)
5.25	0.13575	$24^3 \times 48$	0.080	1.92	0.632(9)
5.29	0.13400	$16^3 \times 32$	0.076	1.21	1.501(13)
5.29	0.13500	$16^3 \times 32$	0.076	1.21	1.094(10)
5.29	0.13550	$24^3 \times 48$	0.076	1.82	0.851(8)
5.29	0.13590	$24^3 \times 48$	0.076	1.82	0.622(6)
5.40	0.13500	$24^3 \times 48$	0.067	1.61	1.184(9)
5.40	0.13560	$24^3 \times 48$	0.067	1.61	0.917(7)
5.40	0.13610	$24^3 \times 48$	0.067	1.61	0.649(5)

simulations at smaller quark masses ongoing

analysis still preliminary!

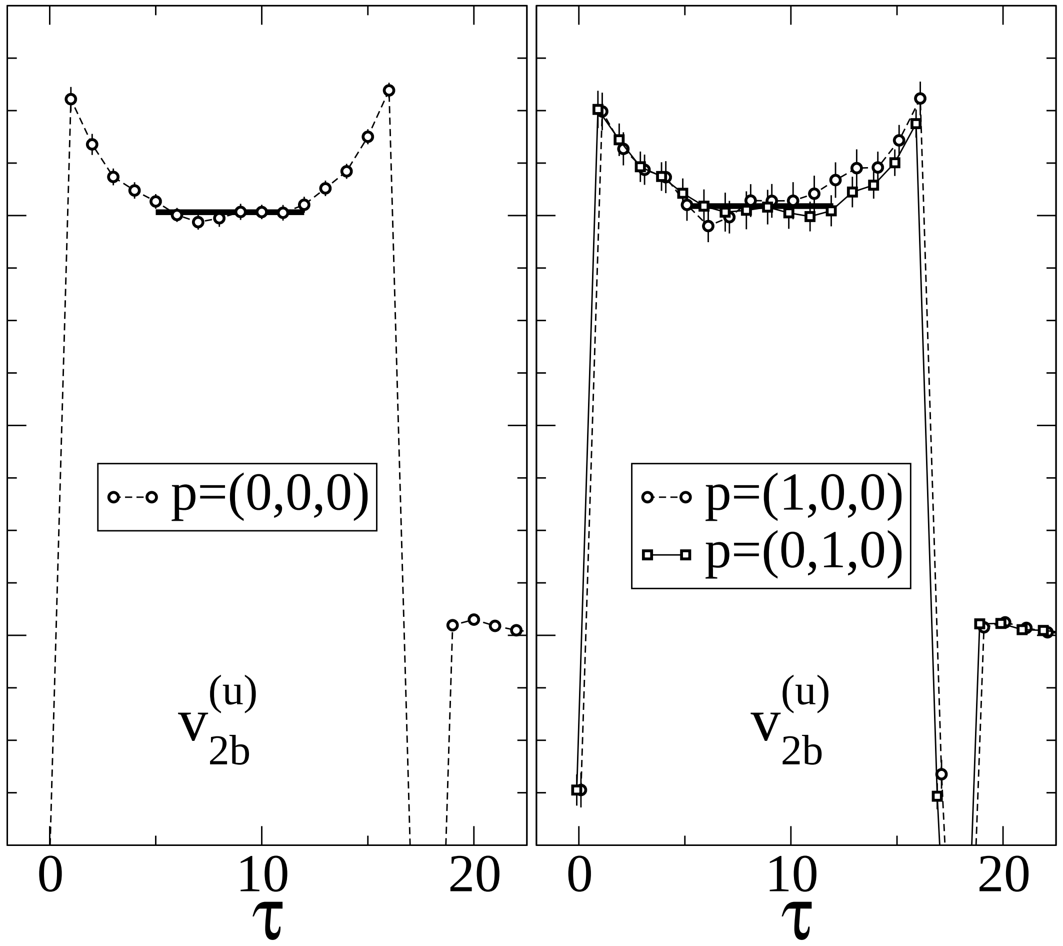
Some raw data

nucleon 2-point function



quenched clover fermions, $\beta = 6.4$, $\kappa = 0.1350$, $32^3 \times 48$ lattice bare ratios $R = \frac{\langle B(t)\mathcal{O}(\tau)\bar{B}(0)\rangle}{\langle B(t)\bar{B}(0)\rangle} = \langle N|\mathcal{O}|N\rangle + \cdots$

for $\langle x\rangle^{(u)} = v_2^{(u)}$ (forward matrix element)



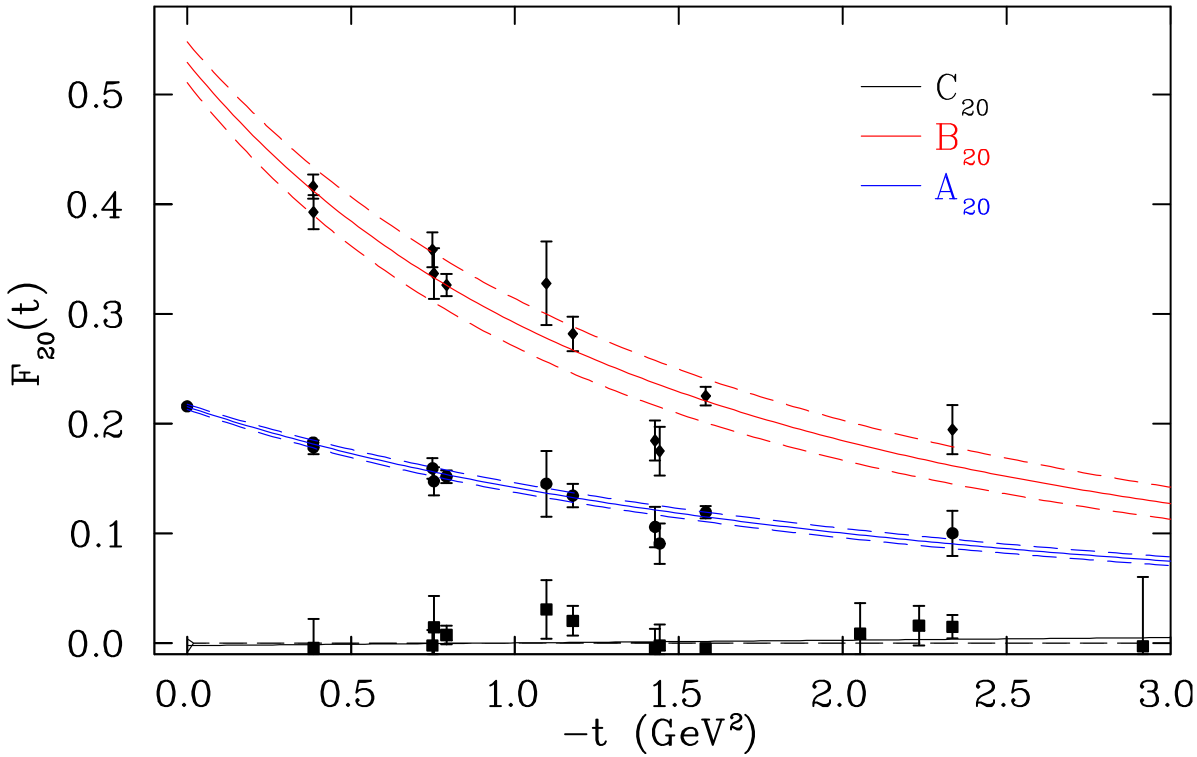
$\beta = 5.4$, $\kappa = 0.1356$, $t/a = 17$

horizontal lines: fit to the data

τ in lattice units

A selection of results

GFFs A_{20}^{u-d}, B_{20}^{u-d}, C_2^{u-d} (non-singlet) (unrenormalised, for $\overline{\text{MS}}$ at $\mu = 2\,\text{GeV}$ multiply by 1.12)



$\beta = 5.4$, $\kappa = 0.1350$

$24^3 \times 48$ lattice

dipole fit:

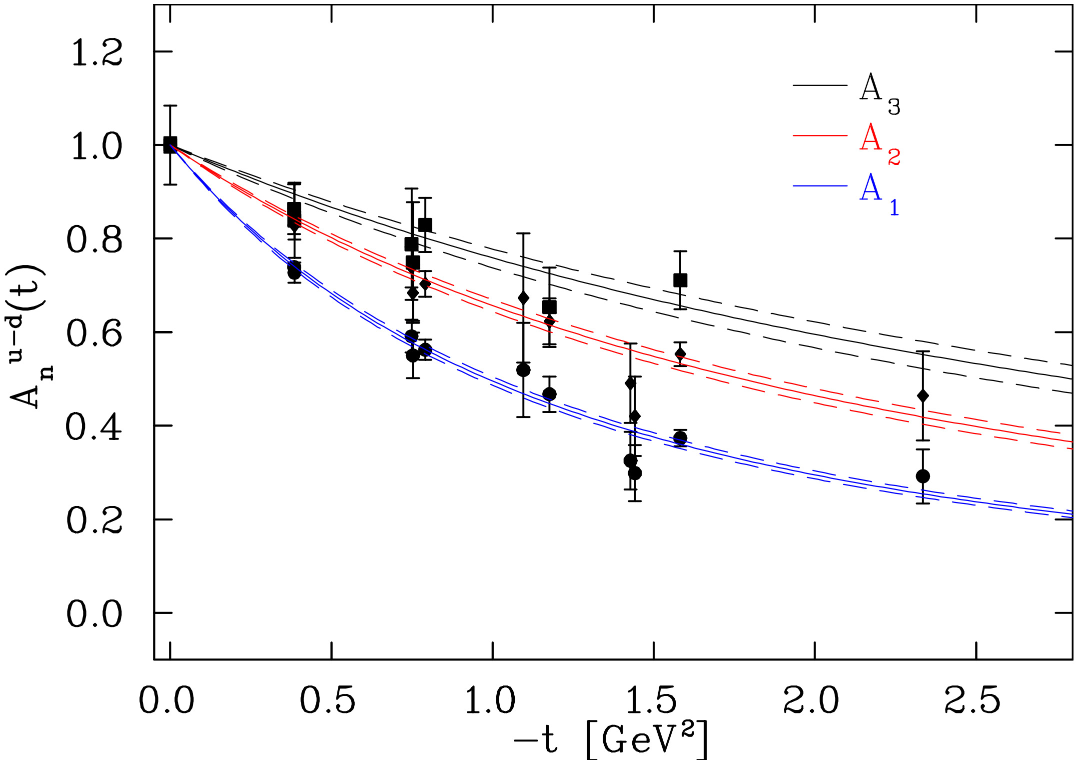
$$A_{n0}(t) = \frac{A_{n0}(0)}{(1 - t/M_n^2)^2}$$

$$= \frac{\langle x^{n-1}\rangle_q}{(1 - t/M_n^2)^2}$$

$$\int_{-1}^{1} \mathrm{d}x\, x H_q(x,\xi,t) = A_{20}^q(t) + 4\xi^2 C_2^q(t) \qquad \int_{-1}^{1} \mathrm{d}x\, x E_q(x,\xi,t) = B_{20}^q(t) - 4\xi^2 C_2^q(t)$$

C_2 practically zero

GFFs $A_{10}^{u-d} = F_1^{u-d}(t)$, A_{20}^{u-d}, A_{30}^{u-d} (non-singlet), normalised to unity at $t = 0$ (dipole fits)



$\beta = 5.4$, $\kappa = 0.1350$
$24^3 \times 48$ lattice

dipole fit:

$$A_{n0}(t) = \frac{A_{n0}(0)}{(1 - t/M_n^2)^2}$$

form factor $A_{n0}(t)$ flattens as n grows
$\leftrightarrow$ dipole mass M_n grows with n

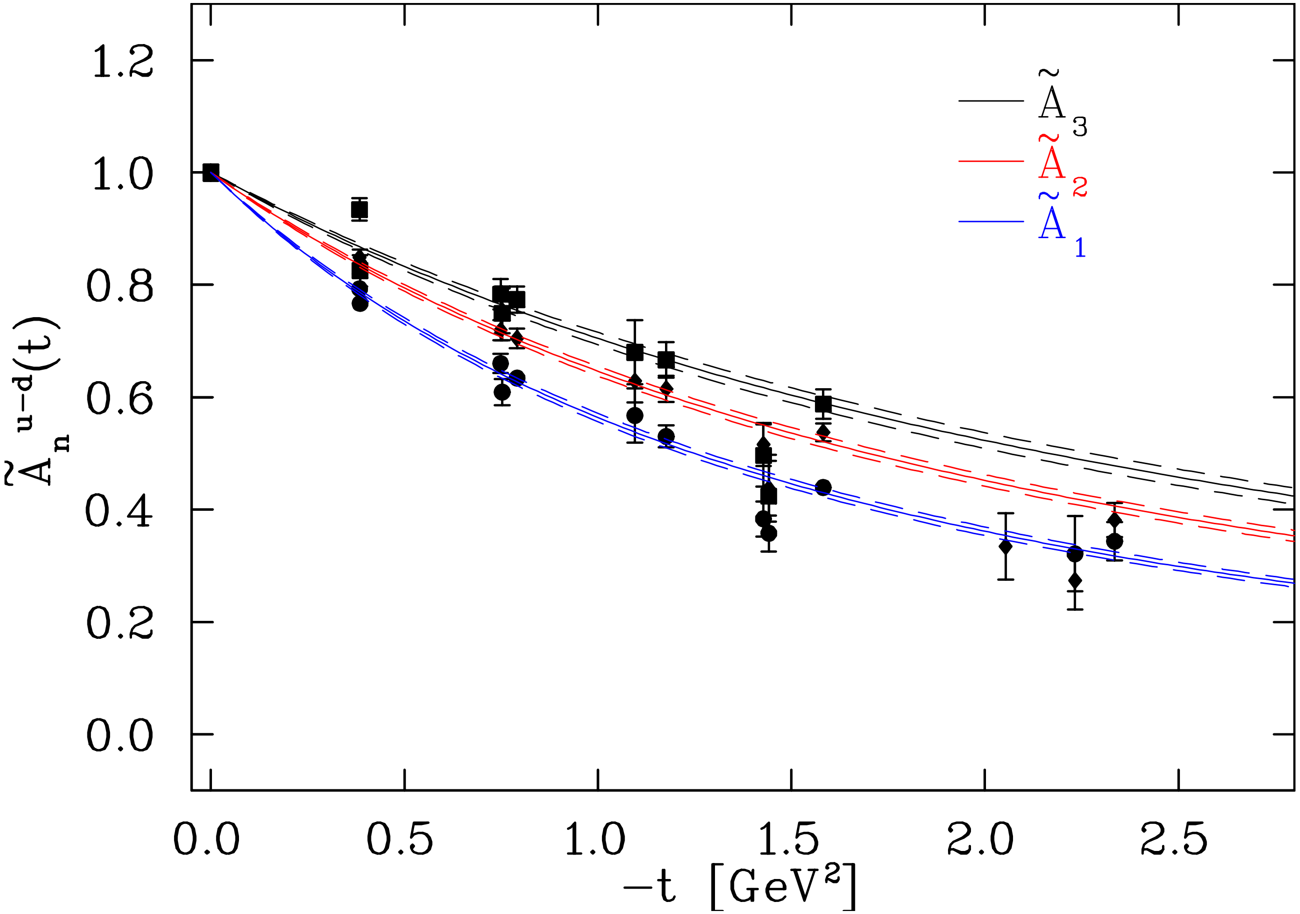
$$\int_{-1}^{1} \mathrm{d}x\, x^{n-1} H_q(x, \xi = 0, t) = A_{n0}^q(t)$$

H_q (as a function of t) becomes wider as x grows

$$q(x, \vec{b}_\perp) = \int \frac{\mathrm{d}^2\Delta_\perp}{(2\pi)^2}\, \mathrm{e}^{\mathrm{i}\vec{b}_\perp \cdot \vec{\Delta}_\perp} H_q(x, 0, -\vec{\Delta}_\perp^2)$$

q (as a function of $\vec{b}_\perp$) becomes narrower as x grows (as expected) polarised case:

generalised form factors $\tilde{A}_{10}^{u-d}$, $\tilde{A}_{20}^{u-d}$, $\tilde{A}_{30}^{u-d}$ (non-singlet) normalised to unity at $t = 0$
($\beta = 5.4$, $\kappa = 0.1350$, $24^3 \times 48$ lattice)

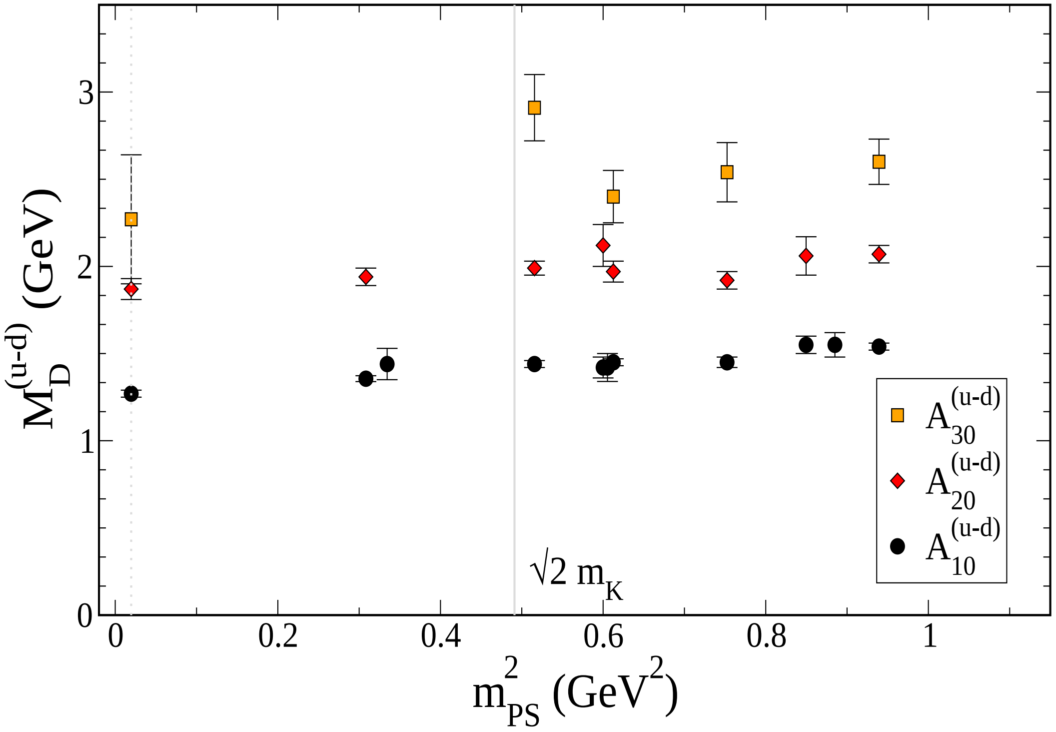


change in slope smaller than in the unpolarised case apparent linearity of the dipole masses as a function of m_π^2

$\longrightarrow$ extend the dipole ansatz $A^q_{n0}(t) = \dfrac{A^q_{n0}(0)}{(1 - t/M_n^2)^2}$

to include also the m_π dependence: $M_n \longrightarrow M_n + \alpha_n m_\pi^2$

$\uparrow$ dipole mass in the chiral limit



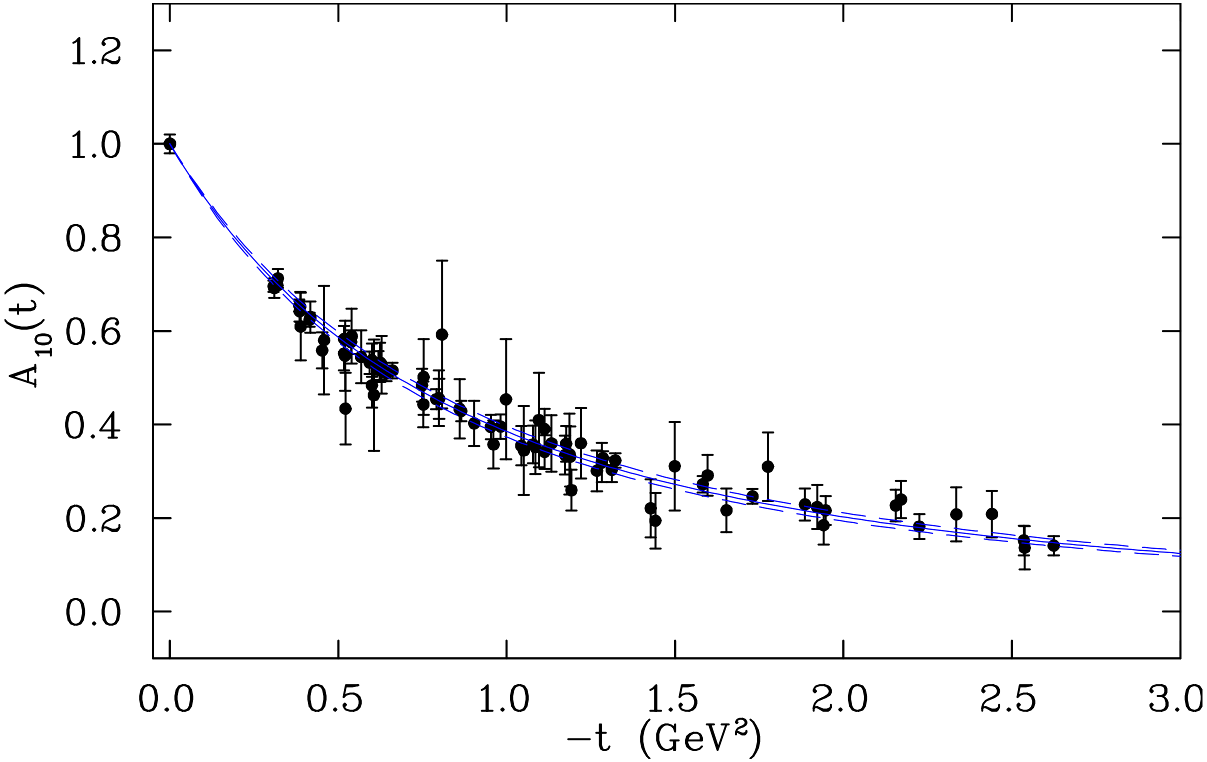
values in the chiral limit from the fit ansatz

$$A^q_{n0}(t) = \frac{A^q_{n0}(0)}{(1 - t/(M_n + \alpha_n m_\pi^2)^2)^2}$$

shift the data to a common curve

$$A^q_{n0}(t) = \frac{A^q_{n0}(0)}{\left(1 - t/(M_n + \alpha_n m_\pi^2)^2\right)^2}$$

at the physical pion mass

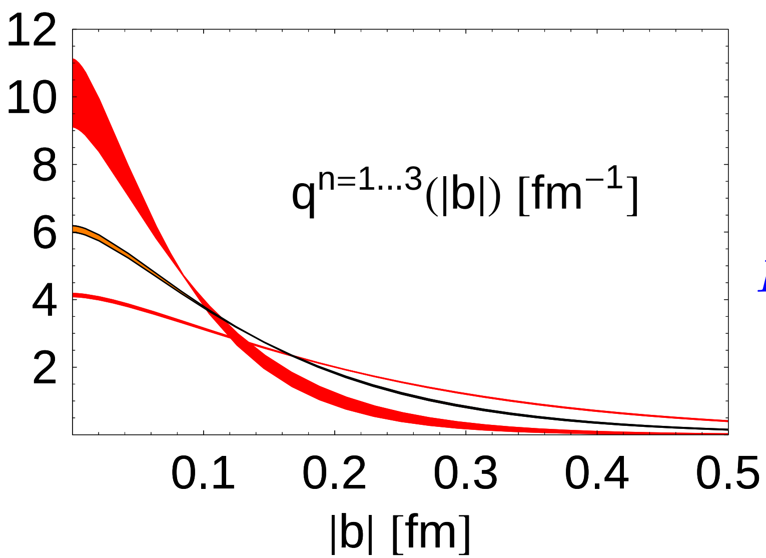


flavour $u - d$
normalised to 1 at $t = 0$

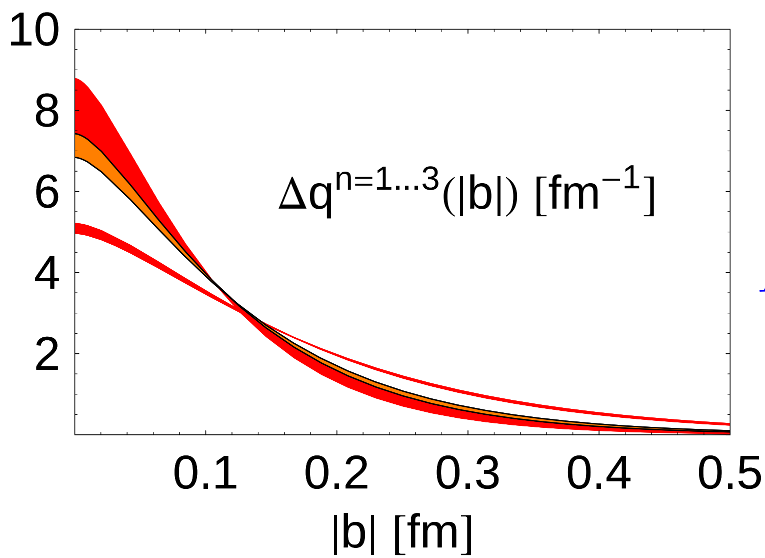
M. G. et al., hep-lat/0501029

lowest three moments of $H(x, \xi = 0, t)$ and $\tilde{H}(x, \xi = 0, t)$
Fourier transform to impact parameter space
with the help of the dipole ansatz extrapolated linearly to the chiral limit:

$$\int \frac{\mathrm{d}^2\Delta_\perp}{(2\pi)^2}\, \mathrm{e}^{\mathrm{i}\vec{b}_\perp\cdot\vec{\Delta}_\perp} \int_{-1}^{1} \mathrm{d}x\, x^{n-1} H_q(x, 0, -\vec{\Delta}_\perp^2)$$
$$= \int \frac{\mathrm{d}^2\Delta_\perp}{(2\pi)^2}\, \mathrm{e}^{\mathrm{i}\vec{b}_\perp\cdot\vec{\Delta}_\perp} \quad \frac{A^q_{n0}(0)}{\left(1 + \vec{\Delta}_\perp^2/M_n^2\right)^2} \quad = \quad \int_{-1}^{1} \mathrm{d}x\, x^{n-1} q(x, \vec{b}_\perp)$$



H



$\tilde{H}$

larger n corresponds to a narrower distribution

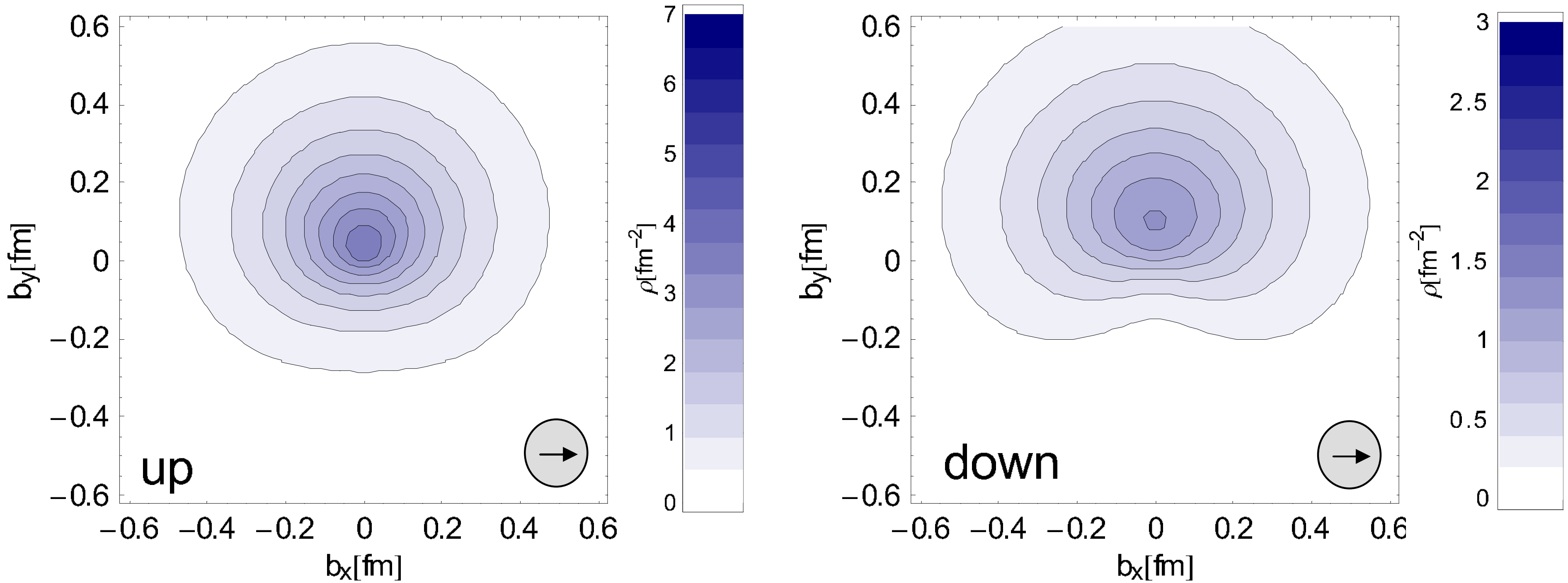
flavour $u - d$

M. G. et al., hep-lat/0609001

what about the GPDs (GFFs) connected with the operators $(\mathrm{i}/2)^n \; \bar{q}\mathrm{i}\sigma_{\mu\nu}\overleftrightarrow{D}_{\mu_1} \cdots \overleftrightarrow{D}_{\mu_n} q$?

$\longrightarrow$ (moments of) the density of transversely polarised quarks in a (possibly also transversely polarised) nucleon in impact parameter space

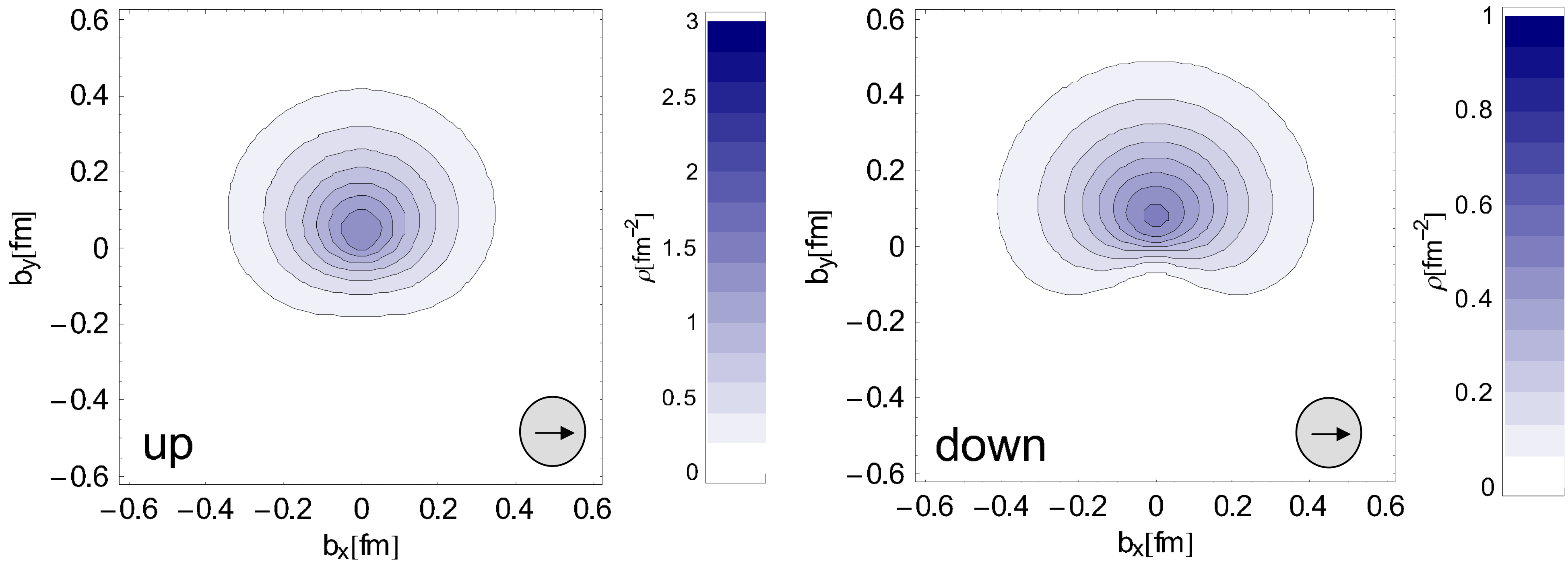
(generalised) dipole parametrisation of the GFFs + linear chiral extrapolation



x^0 moment, nucleon unpolarised

distortion in positive y-direction for u and d quarks

x^1 moment qualitatively similar, but more strongly peaked:



more detailed interpretation and conjectured relation to phenomenology:
M. Burkardt, Phys. Rev. D72, 094020 (2005)

Summary and outlook

- The structure of the nucleon remains an exciting subject, both experimentally and theoretically.
  QCD is a rather non-trivial theory.
- Monte Carlo simulations of lattice QCD $\longrightarrow$ large distance properties
  very active field with quite a few collaborations
- Results from simulations with $N_f = 2$ dynamical clover fermions (QCDSF together with UKQCD)
- QCDSF-UKQCD compute matrix elements which allow the evaluation of moments of GPDs
- The data reveal the expected "shrinking" of the nucleon for $x \to 1$.
- Three-dimensional picture of the nucleon seems possible

for the future:

- Simulations on larger lattices — for smaller momenta in the (generalised) form factors
- Simulations for smaller quark masses — to make the chiral extrapolation more reliable