

# Lattice QCD

## at finite temperature and chemical potential

### Introduction

- I Phase diagram at  $\mu = 0$
- II Equation of state at  $\mu = 0$
- III Phase diagram at  $\mu \neq 0$
- IV Equation of state at  $\mu \neq 0$

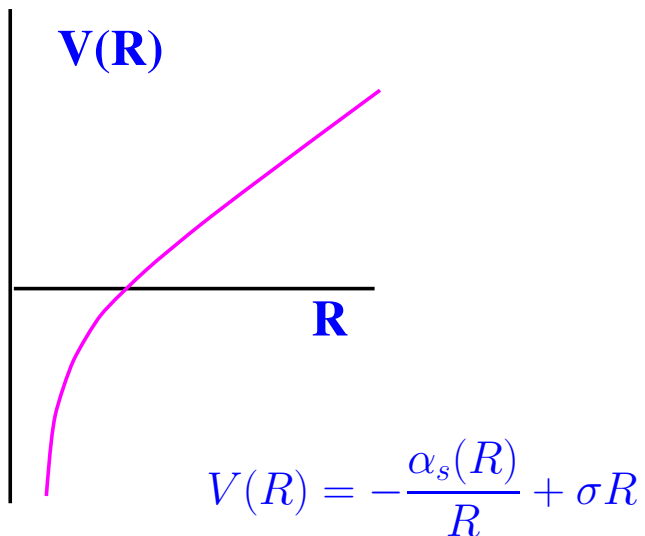
# Introduction

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non perturbative phenomena in the hadron phase :

## Confinement

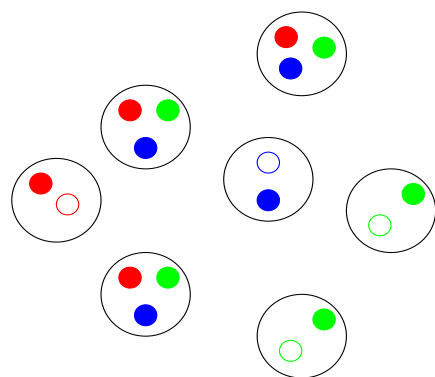
- quarks and gluons are permanently confined in hadrons
- heavy quarks : potential models



## spontaneous breaking of chiral symmetry

- for  $m_f = 0$  :  $\mathcal{L}_{QCD}$  invariant under
$$q_R \rightarrow U_R q_R \quad U_R \in SU_R(N_F)$$
$$q_L \rightarrow U_L q_L \quad U_L \in SU_L(N_F)$$
- $m_u, m_d \leq 10 \text{ MeV} \ll m_{proton} \simeq 1000 \text{ MeV}$   
 $m_s \simeq 100 \text{ MeV}$   
 $m_c \simeq 1.3 \text{ GeV}$   
 $m_b \simeq 4.5 \text{ GeV}$   
 $m_t \simeq 175 \text{ GeV}$
- vacuum invariant  
 $\Rightarrow$  parity doubling in the mass spectrum or
- spontaneous breaking  
 $\Rightarrow$  Goldstone particles  $m_\pi \simeq 140 \text{ MeV}$   
 $\Rightarrow$  chiral condensate  $\langle \bar{q}q \rangle \neq 0$

at high temperatures and/or density : transition to a new state of matter

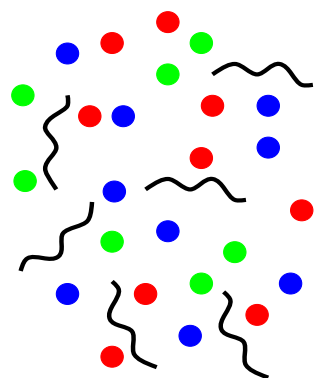


nuclear matter

$$T < T_c$$



$$T_c \cong 1/\text{fm} \cong 200 \text{ MeV}$$



quark gluon plasma

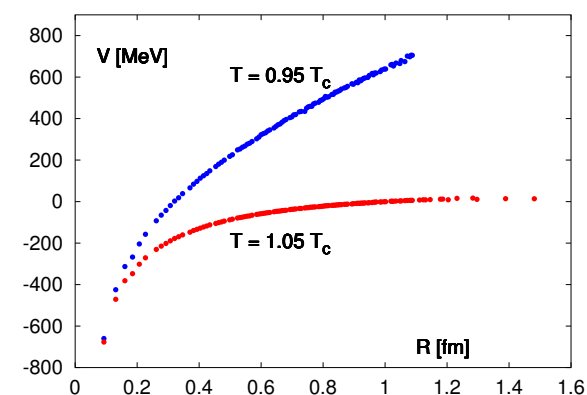
$$T > T_c$$

**Deconfinement**

$$V(R) \sim \frac{1}{R} e^{-m_E R}$$

**restoration of chiral symmetry**

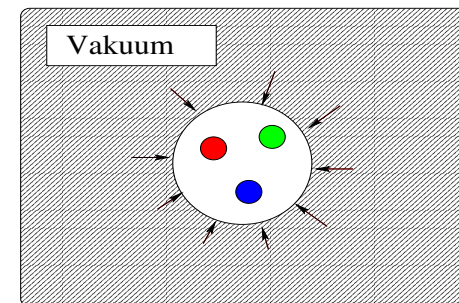
$$\langle \bar{q}q \rangle = 0$$



expected properties - qualitatively :

bag model

vacuum pressure  $p = B$



ideal gas (Stefan Boltzmann)

hadron - view

$$p = d_H \frac{\pi^2}{90} T^4$$

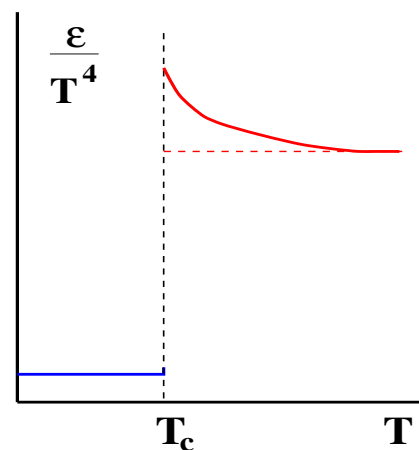
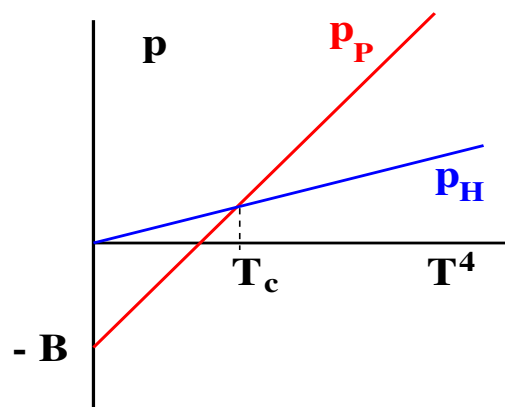
$$d_H = \# \text{ d.o.f.} = 3 \quad (\pi^\pm, \pi^0)$$

quark - gluon - view

$$p = d_P \frac{\pi^2}{90} T^4 - B$$

$$d_P = 2 \times 8 + 21/2 N_F \quad (G, q)$$

$F/V = -p$  minimal  
 $\Rightarrow$  phase transition



$$\frac{\epsilon_H}{T^4} = d_H \frac{\pi^2}{30}$$

$$\frac{\epsilon_P}{T^4} = d_P \frac{\pi^2}{30} + \frac{B}{T^4}$$

- $p_H(T_c) = p_P(T_c) \Leftrightarrow T_c = \left( \frac{90}{\pi^2(d_P - d_H)} \right)^{1/4} B^{1/4} \simeq 0.7 B^{1/4} \simeq 140 \text{ MeV}$

- $\epsilon_P(T_c) \simeq 15 T_c^4 \simeq 1 \text{ GeV}/fm^3$

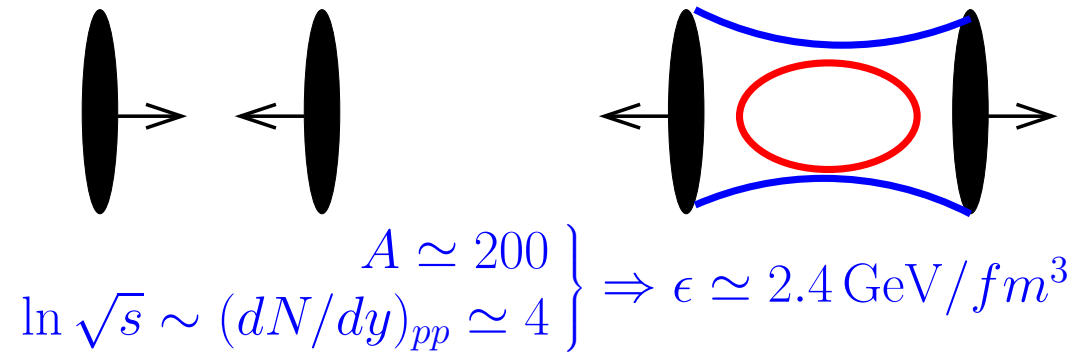
- $t_c \simeq \frac{2.4}{\sqrt{d(T_c)}} \left( \frac{1 \text{ MeV}}{T_c} \right)^2 \text{ sec} \simeq 10^{-5} \text{ sec}$

→ early universe

- $\epsilon_{\text{plasma}}(T_c) \simeq 4 \epsilon_{\text{neutron}} \simeq 1 \text{ GeV}/fm^3$

→ neutron stars

- → heavy ion collisions (RHIC, LHC)  
 $\epsilon \simeq 0.1 A^{1/3} \left( \frac{dN}{dy} \right)_{pp} [\text{GeV}/fm^3]$



## Questions to theory :

- phase diagram
  - critical temperature  $T_c$
  - nature of the transition:  
order, critical exponents
- Equation of State  $\epsilon(T), p(T)$ 
  - critical energy density
  - latent heat
- properties of the plasma phase
  - existence of hadronic excitations
  - masses, widths
  - screening lengths
  - strangeness content
  - response functions
  - ...
- properties of the hadron phase at  $T \lesssim T_c$ 
  - “approach to chiral symmetry”  
→ hadron masses and widths
  - “approach to deconfinement”  
→ quark potentials
  - ...

→ **genuine non-perturbative methods required**

Quantum Statistics in equilibrium :

$$\text{partition function } Z = \text{tr} \left\{ e^{-\hat{H}/T} \right\}$$

→ **Feynman path integral**

$$Z(T, V) = \int \mathcal{D}\phi(\vec{x}, \tau) \exp \left\{ - \int_0^{1/T} d\tau \int_0^V d^3\vec{x} \mathcal{L}_E[\phi(\vec{x}, \tau)] \right\}$$

- integral over all configurations  $\phi(\vec{x}, \tau)$
- euclidean “time”  $\tau = it$
- weighted by Boltzmann factor  $\exp(-S_E)$
- (anti-) periodic boundary conditions in  $\tau$

apply standard thermodynamic relations, e.g.

$$\text{energy density} \quad \epsilon = \frac{T^2}{V} \frac{\partial \ln Z}{\partial T} \Big|_V$$

$$\text{specific heat} \quad c_V = \frac{1}{VT^2} \frac{\partial^2 \ln Z}{\partial (1/T)^2} \Big|_V$$

in general

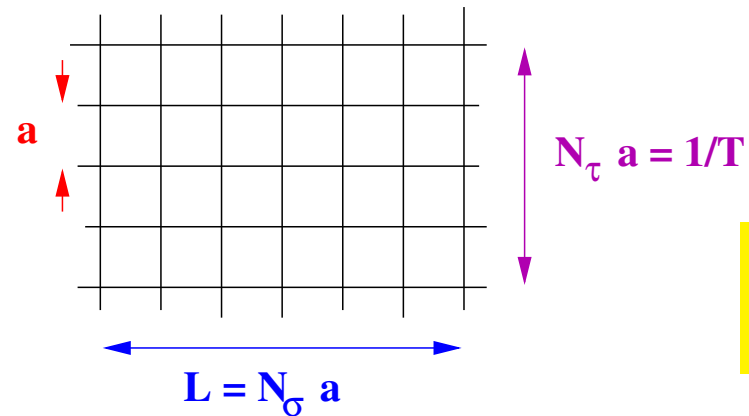
$$\langle \mathcal{O} \rangle = \frac{1}{Z} \text{tr} \left\{ \hat{\mathcal{O}} e^{-\hat{H}/T} \right\} = \frac{1}{Z} \int \mathcal{D}\phi \mathcal{O}[\phi] e^{-S_E[\phi]}$$

also : starting point of perturbation theory i.e. expansion in coupling strength  $g$

numerical treatment of QCD  $\Rightarrow$  discretize (Euclidean) space-time

$\Rightarrow$  **lattice**

$$N_\sigma^3 \times N_\tau$$



$$Z(T, V) = \int \prod_{i=1}^{N_\tau N_\sigma^3} d\phi(x_i) \exp \{-S[\phi(x_i)]\}$$

finite yet high-dimensional path integral

$\rightarrow$  **Monte Carlo**

• thermodynamic limit, IR - cut-off effects

• continuum limit, UV - cut-off effects

• chiral limit

numerical effort  $\sim (1/m)^p$

$$LT = \frac{N_\sigma}{N_\tau} \rightarrow \infty$$

$$aT = \frac{1}{N_\tau} \rightarrow 0$$

$$m \rightarrow m_{\text{phys}} \simeq 0$$

# I. Phase Diagram at $\mu = 0$

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Localisation of the phase transition

- **order parameter**

1.) **Polyakov loop**  $L(\vec{x}) = \frac{1}{3} \text{tr} \prod_{\tau} U_{\tau}(\vec{x}, \tau)$

- sensitive on  $Z(3)$  symmetry (pure gauge theory only)
- measures free energy of an isolated quark  $\langle L \rangle \sim e^{-F_{\text{quark}}/T}$

hadron phase	$F_{\text{quark}} \rightarrow \infty$
	$\langle L \rangle = 0$
plasma phase	$F_{\text{quark}} \text{ finite}$
	$\langle L \rangle \neq 0$

2.) **chiral condensate**

- sensitive on chiral symmetry ( $m \rightarrow 0$ )

hadron phase	$\langle \bar{q}q \rangle \neq 0$
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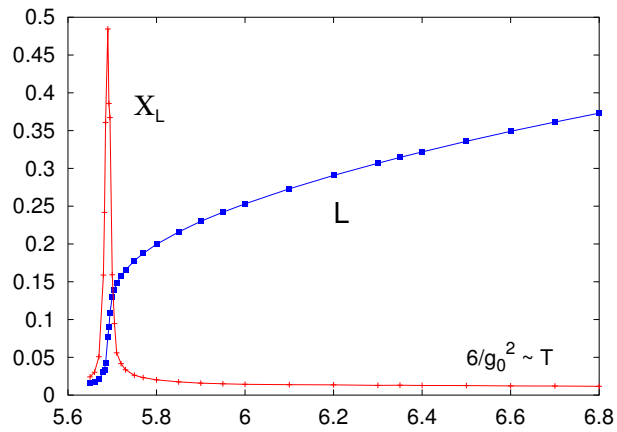
plasma phase	$\langle \bar{q}q \rangle = 0$
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- **susceptibilities**

- e.g. chiral susceptibility
- measures fluctuations

$$\chi_m \sim \frac{\partial^2 \ln Z}{\partial m^2} \sim \langle (\bar{q}q)^2 \rangle - \langle \bar{q}q \rangle^2$$

quenched QCD

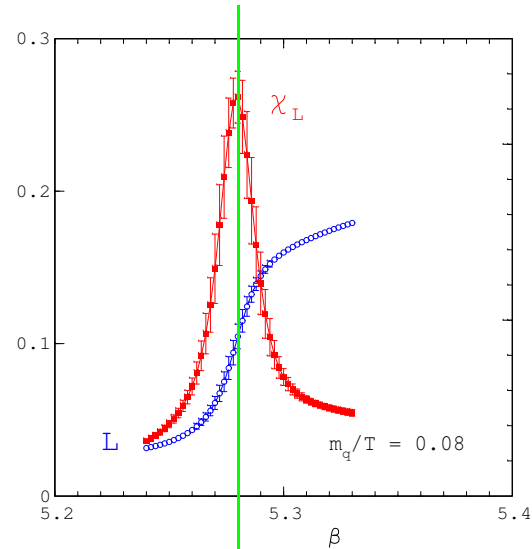


$$T = \frac{1}{N_\tau a(g)}$$

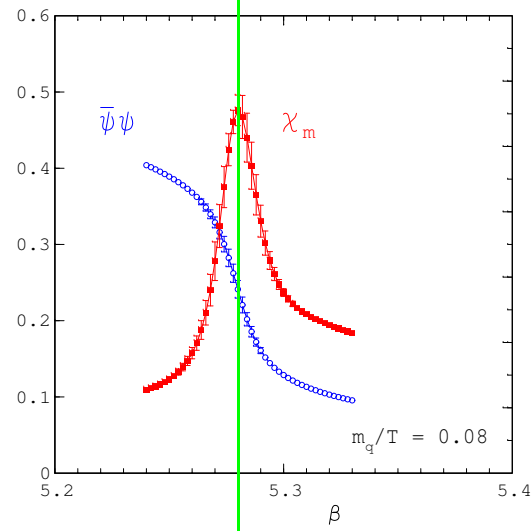
in perturbation theory (lowest order)

$$a = \frac{1}{\Lambda} \exp \left\{ -\frac{24\pi^2}{33 - 2N_F} \frac{1}{g^2} \right\}$$

full QCD ( $N_F = 2$ )



Polyakov loop



chiral condensate

$$g_{crit}^{Polyakov} = g_{crit}^{chiral}$$

$$\beta = 6/g^2$$

critical temperature  $T_c$

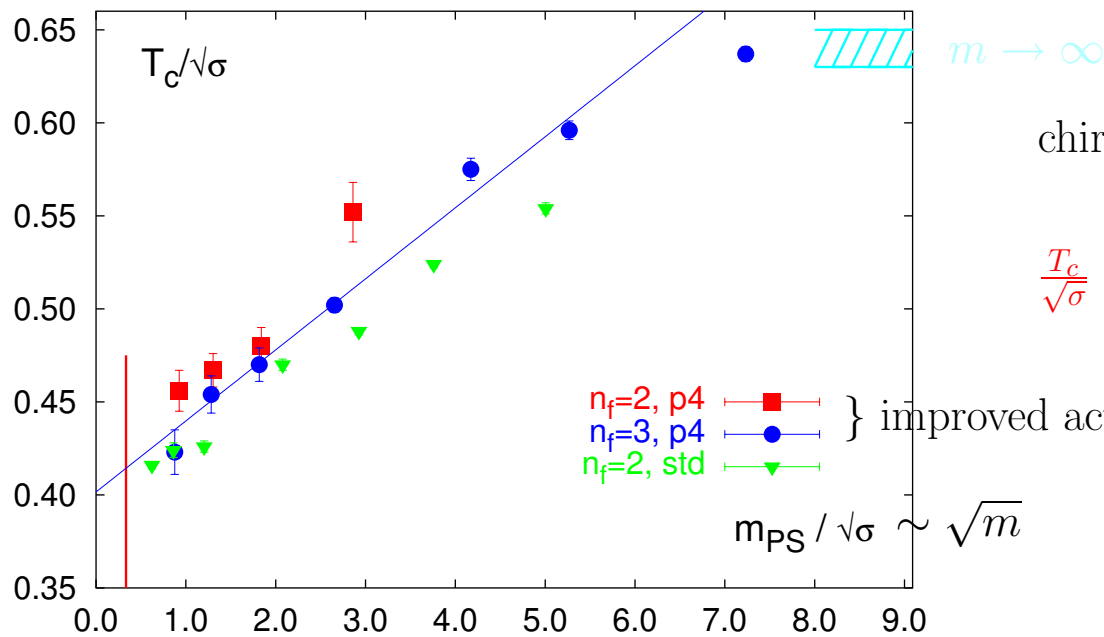
$$T_c = \frac{1}{N_\tau a(g_c)}$$

at  $T = 0$ , same (bare) coupling  $g_c$ , measure e.g. string tension  $\sigma \Rightarrow: \sqrt{\sigma} a(g_c) = \text{number}$

$\Rightarrow$  dimension less ratio

$$\frac{T_c}{\sqrt{\sigma}} = \frac{1}{N_\tau a(g_c)} \star \frac{a(g_c)}{\text{number}} = \frac{1}{N_\tau \star \text{number}}$$

$\sqrt{\sigma}$  only weakly affected by quark mass



chiral extrapolation  $m \rightarrow 0$

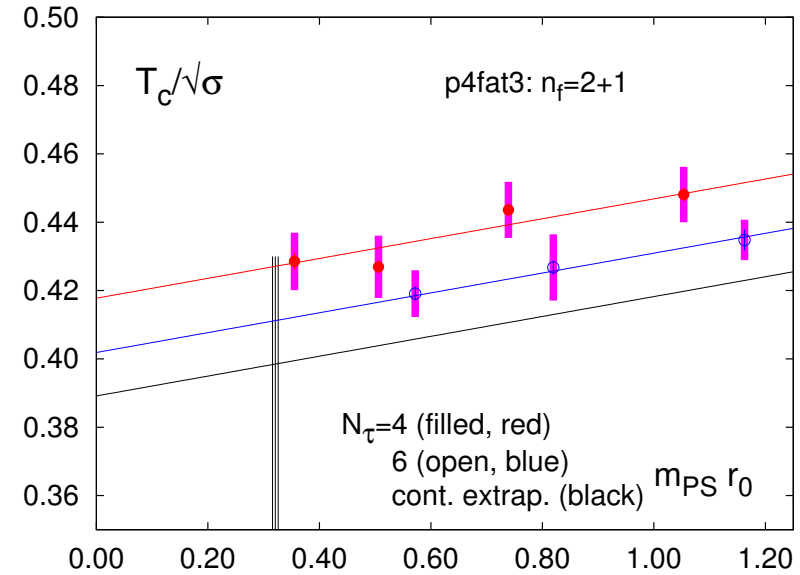
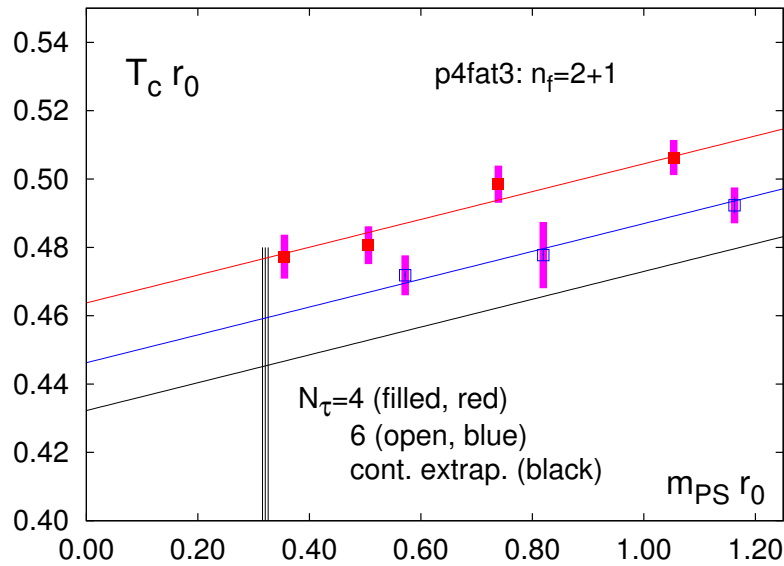
$$\frac{T_c}{\sqrt{\sigma}} \rightarrow \begin{cases} 0.42 (\pm 0.01) \text{ (syst.)} & N_F = 2 \text{ Wilson} \star \\ 0.42 (\pm 0.01) \text{ (syst.)} & N_F = 2 \text{ staggered} \\ 0.40 (\pm 0.01) \text{ (syst.)} & N_F = 3 \text{ staggered} \end{cases}$$

[cp-pacs collaboration]

Caveat: not yet continuum limit

but: improved actions

new results:  $N_F = 2 + 1$ ,  $N_\tau = 4, 6$ , exact algorithm (RHMC)



combined continuum/chiral extrapolation ( $d = 1.08$  for  $O(4)$ ,  $d = 2$  for first order)

$$(T_c r_0)_{m_l, N_\tau} = T_c r_0 + A(m_{PS} r_0)^d + B/N_\tau^2$$

chiral limit  $T_c r_0 = 0.444(6)_{-2}^{+12}$   $T_c / \sqrt{\sigma} = 0.399(5)_{-1}^{+10}$

phys. point  $T_c r_0 = 0.456(7)_{-1}^{+3}$   $T_c / \sqrt{\sigma} = 0.408(7)_{-1}^{+3}$

with new  $T = 0$  MILC (lattice) results for  $r_0 = 0.469(7)\text{fm}$  obtain:  $T_c = 192(5)(4)\text{MeV}$

## nature of the phase transition (at $\mu = 0$ )

expected

- $\Phi(\tau) = \sum_n \exp\{i\omega_n \tau\} \Phi(\omega_n)$  with Matsubara frequencies  $\omega_n = \begin{cases} 2\pi T n & \text{bosons} \\ \pi T(2n + 1) & \text{fermions} \end{cases}$ 
  - for high temperatures static boson-modes only
  - three-dimensional effective theory
- long range correlations
  - global symmetries count, microscopic details don't (universality)
  - here: chiral symmetries,  $\sigma$  models

$\Rightarrow N_F = 2$

[Wilczek,Pisarski]

- if phase transition continuous (2nd order), then  $SU_R(2) \otimes SU_L(2) \simeq O(4)$
- if  $U_A(1)$  effectively restored (no non-trivial topological configurations at  $T_c^{chiral}$ ), then phase transition discontinuous (1st order).

$\Rightarrow N_F = 3$

[Wilczek,Pisarski]

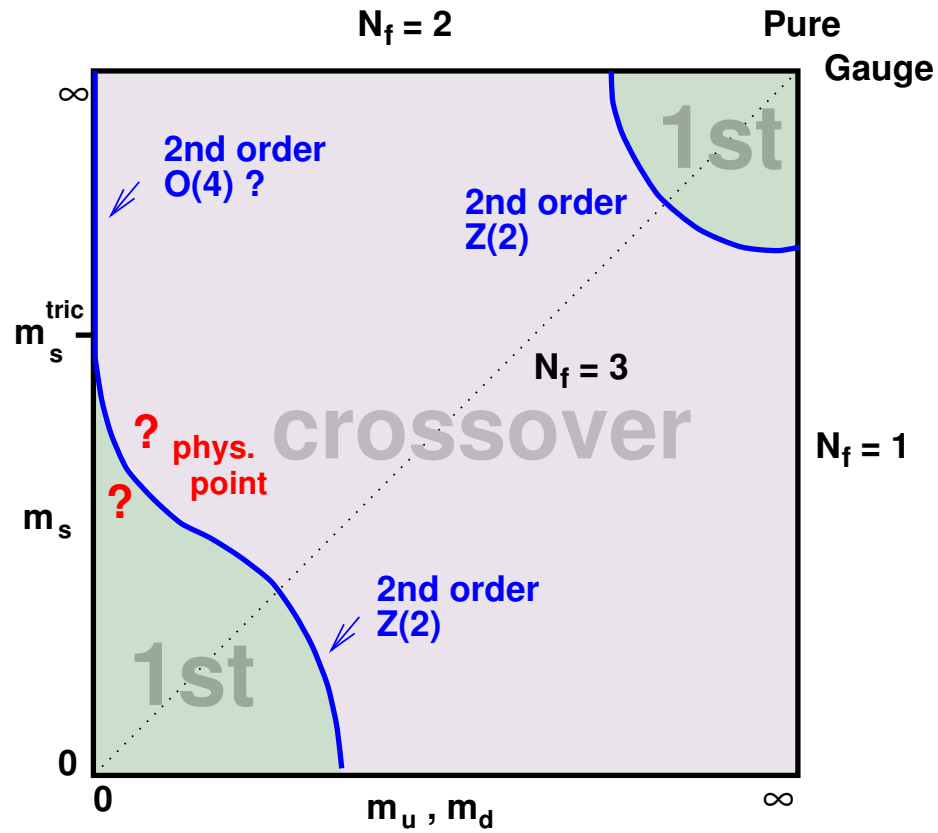
- phase transition discontinuous → even at  $m \leq m_c \neq 0$
- at critical end point  $m_c$ :  $Z(2)$  Ising universality

[Gavin,Gocksch,Pisarski]

$\Rightarrow N_F = 2 + 1$

- depending on quark masses  $m_{u,d}, m_s$

expected phase diagram in the  $m_{u,d} - m_s$  plane ( $\mu = 0$ )



## critical behavior

in the vicinity of a phase transition: correlation length  $\rightarrow \infty$

$\Rightarrow$  **scaling behavior** of the free energy density

$$f(t, m, L) = b^{-d} f(b^{y_t} t, b^{y_h} m, L/b) \quad \text{with reduced temperature } t = \frac{|T-T_c|}{T_c}$$

$\Rightarrow$  **scaling laws**, e.g.

$$\begin{aligned} \langle M \rangle &\sim m^{1/\delta} \\ \chi_m &\sim L^{\gamma/\nu} \\ B_4 &= \frac{\langle (\delta M)^4 \rangle}{\langle (\delta M)^2 \rangle^2} \end{aligned} \quad \text{(here: } \delta M = M - \langle M \rangle, M \simeq \bar{q}q)$$

with **critical exponents**  $\delta, \gamma, \nu, \dots$  und **Binder-cumulant**  $B_4$  **universal**

	Z(2)	O(2)	O(4)
$\gamma/\nu$	1.963(3)	1.962(5)	1.975(4)
$B_4$	1.604(1)	1.242(2)	1.092(3)

$$N_F = 2$$

- conflicting results for critical behavior

- Wilson  $\langle M \rangle$  scales  $\sim O(4)$  in  $m$

[Iwasaki et al.]

- staggered  $\chi_m, \chi_t$  does not scale  $\sim O(4)$  in  $m$

[Karsch,EL; JLQCD; MILC]

- staggered  $\langle M \rangle$  scales  $\sim O(4)$  in  $L$

[Engels et al.]

- staggered  $c_V$  scales as 1st order

[Di Giacomo et al.]

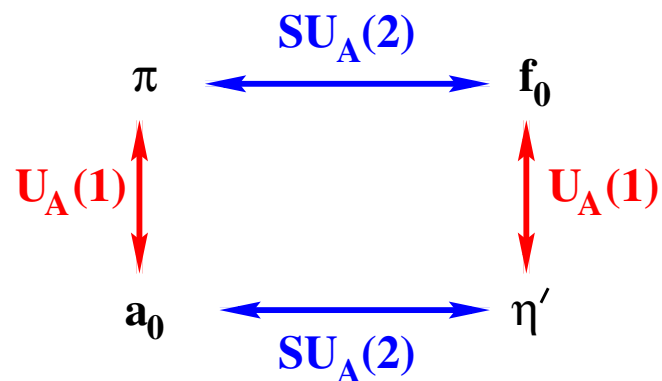
- staggered  $\langle M \rangle$  is as in  $O(2)$  at finite  $L$

[Kogut, Sinclair]

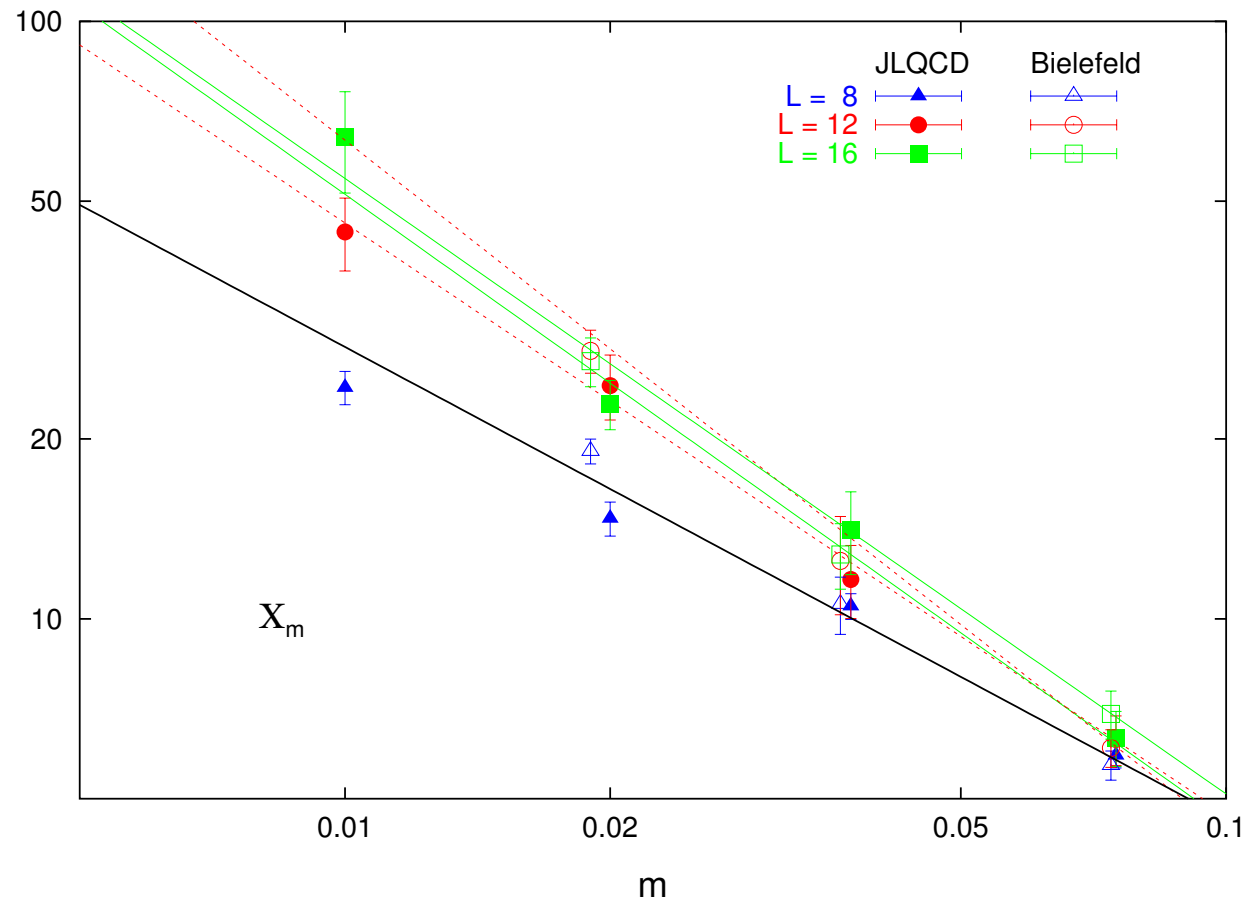
- $U_A(1)$ :

- if effectively restored, then mass spectrum degeneracy

[Shuryak; Cohen et al.; ...]

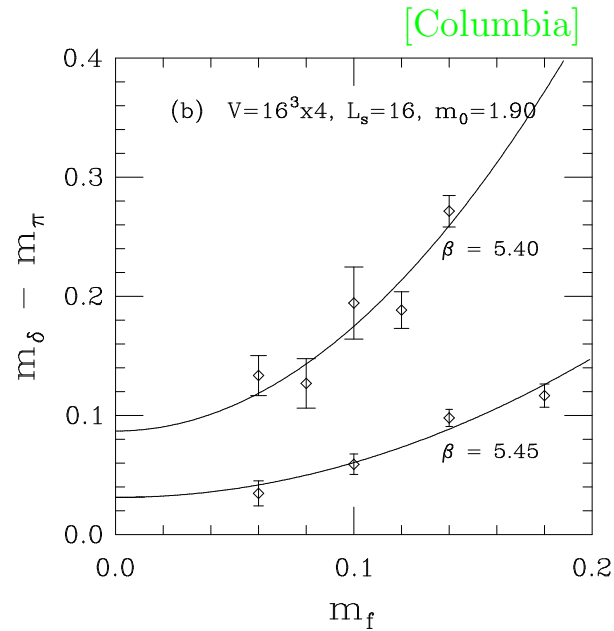


staggered  $\chi_m$  does not scale  $\sim O(4)$  in  $m$  [Karsch, EL; JLQCD]



screening masses  
in accord with  
theoretical expectation

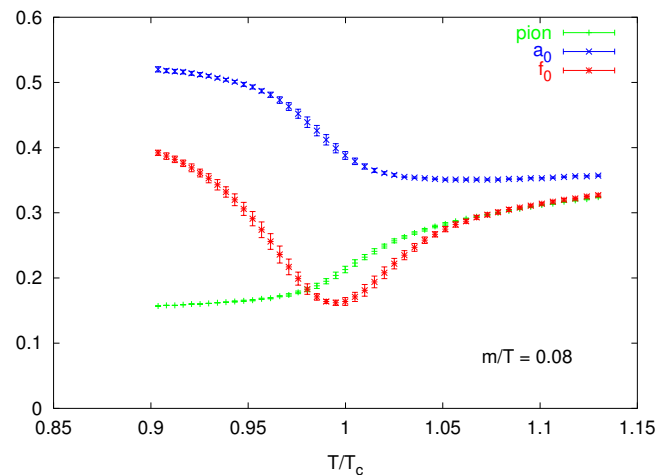
$$m_{a_0} - m_\pi = c_0 + c_2 m^2 + \dots$$



at  $\beta = 5.40 (T \simeq 1.2T_c)$ :  $c_0$  small  $\neq 0$   
 $\Rightarrow U_A(1)$  weakly broken

similarly:  
generalized susceptibilities

$$\chi_H \sim 1/M_H^2$$



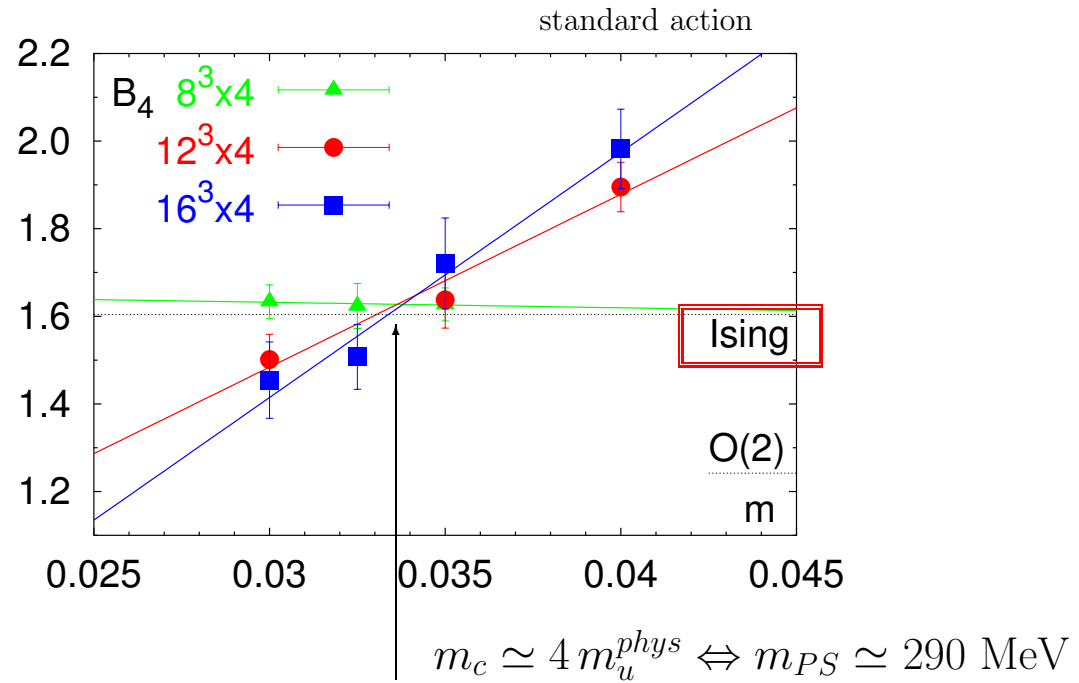
at  $T \simeq T_c$ :

$\chi_{f_0} \simeq \chi_\pi$   $SU_A(2)$  restored  
 $\chi_{a_0} \neq \chi_\pi$   $U_A(1)$  not restored

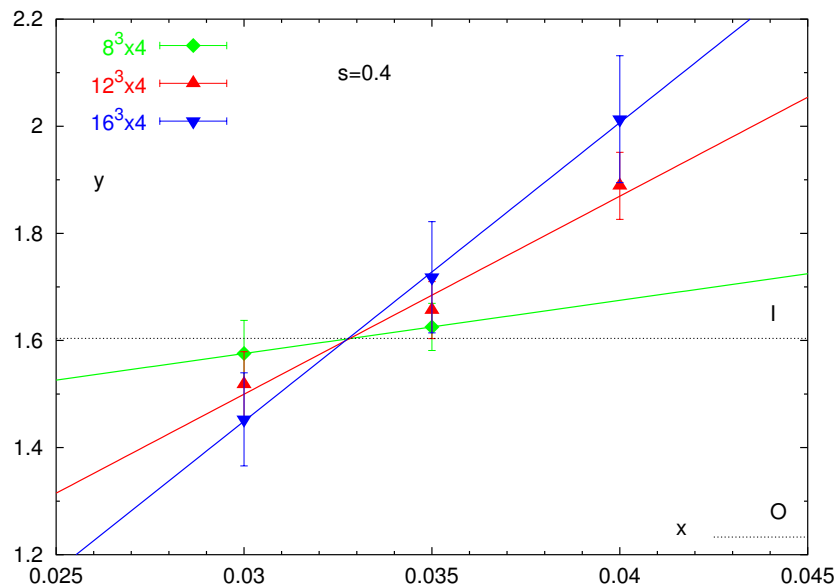
$N_F = 3$

Binder cumulant  $B_4$

- intersection for various  $V$  yields critical value of  $m$
- value of  $B_4$  is universal
- corrections from  $V$  finite and 'order parameter not matched correctly'



[Bielefeld; deForcrand, Philippsen]



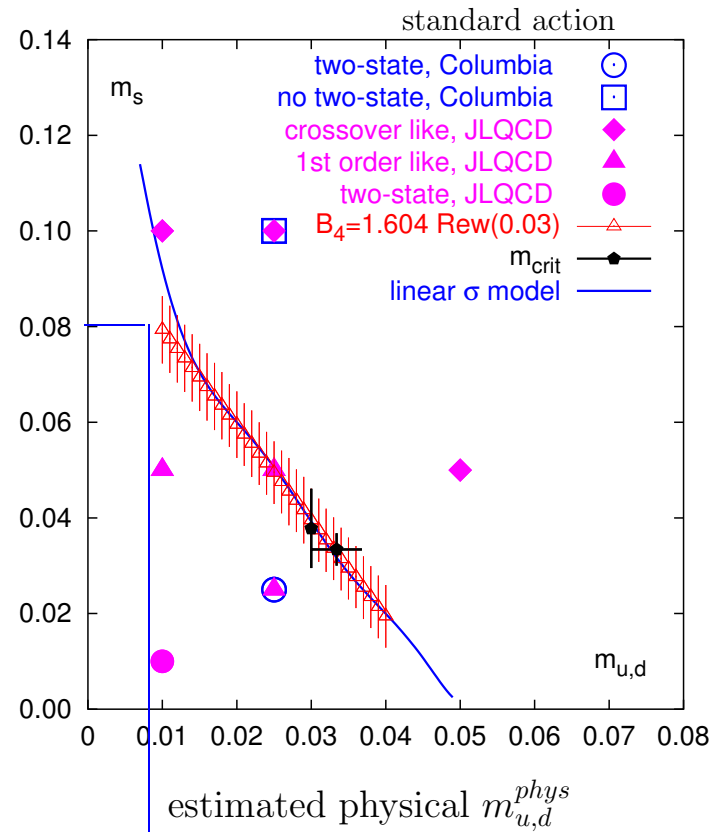
magnetization-like order parameter  $\mathcal{M}$

not identical with chiral condensate  $\langle \bar{q}q \rangle$

(chiral symmetry broken by  $m_q \neq 0$  anyway)

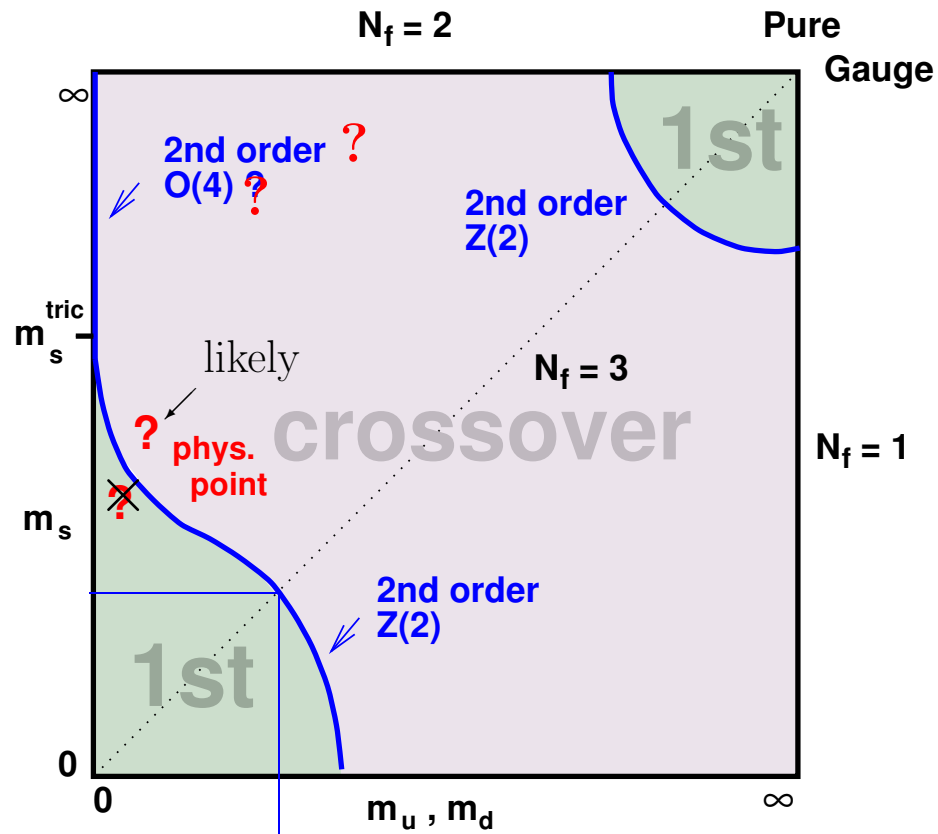
$$\mathcal{M} = \langle \bar{q}q \rangle + s S$$

$N_F = 2 + 1$



$\rightarrow \frac{m_s^{crit}}{m_{u,d}^{phys}} \simeq 10 \quad \text{compare} \quad \frac{m_s^{phys}}{m_{u,d}^{phys}} \simeq 20$

1st order at physical point unlikely

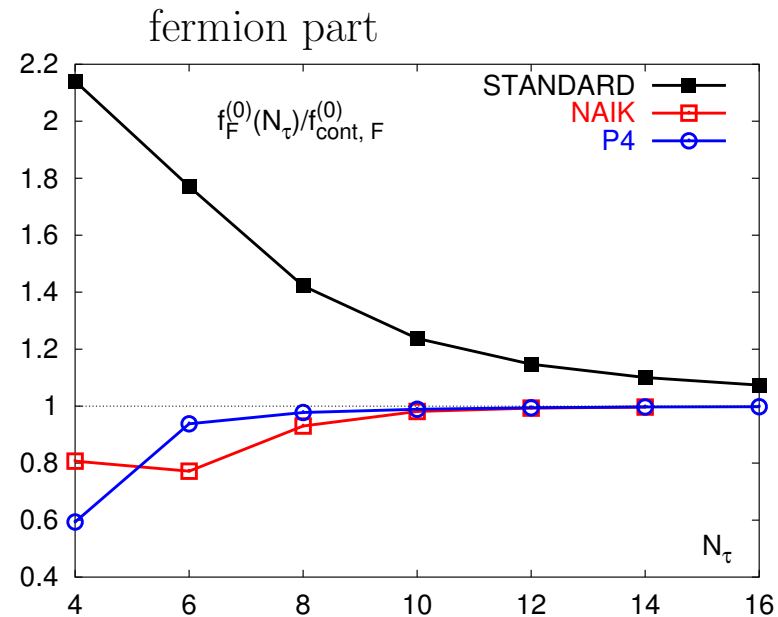
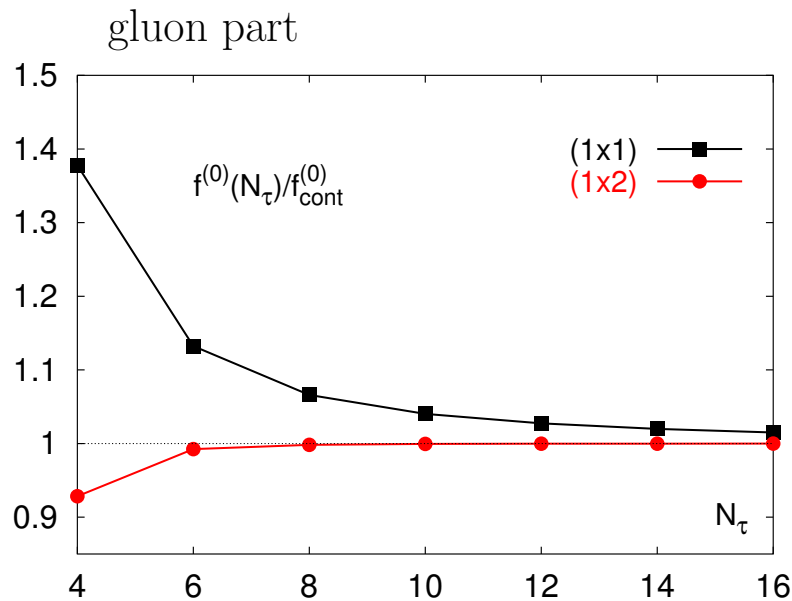


standard  $m_{PS} \simeq 290$  MeV  
 $\rightarrow m_{PS} \simeq 70$  MeV improved

## II. Equation of state at $\mu = 0$

$$\frac{p}{T^4} = N_\tau^4 \times \{\text{signal} \pm \text{statistical error}\}$$

ideal gas limit at finite  $aT = 1/N_\tau$ :

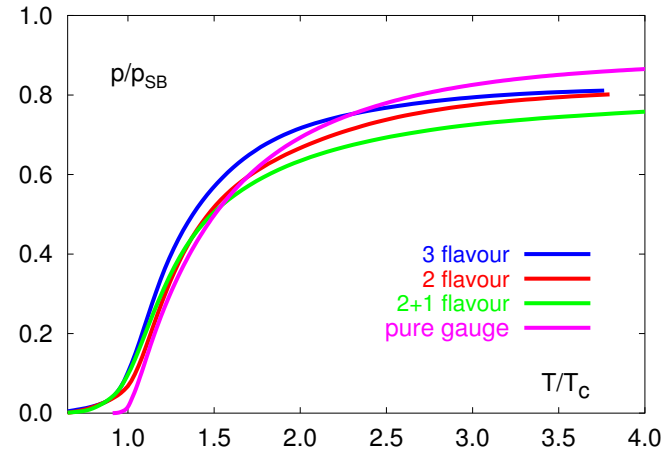
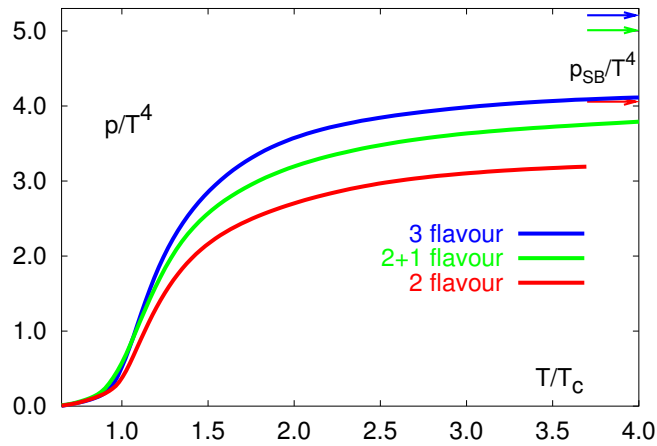


⇒ improved actions mandatory

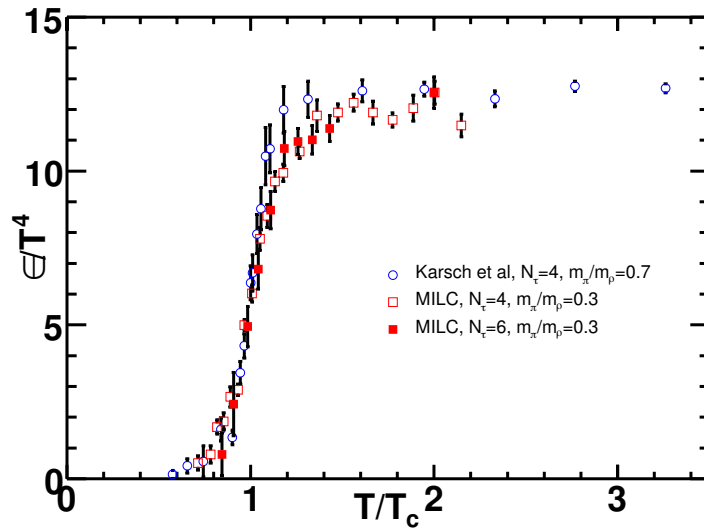
improving thermodynamics: Naik action, p4 action

improving flavor symmetry: fat links, stout links

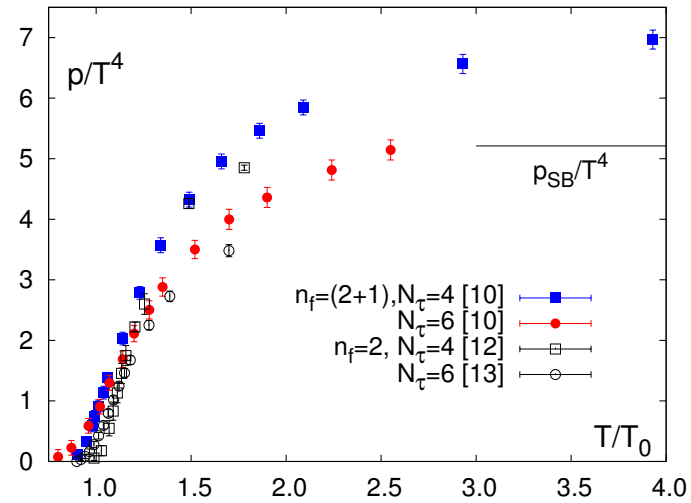
- old results:  $16^3 \times 4$ ,  $m_\pi/m_\rho \simeq 0.7$  [Karsch, EL, Peikert]



- new results: MILC, LAT2005



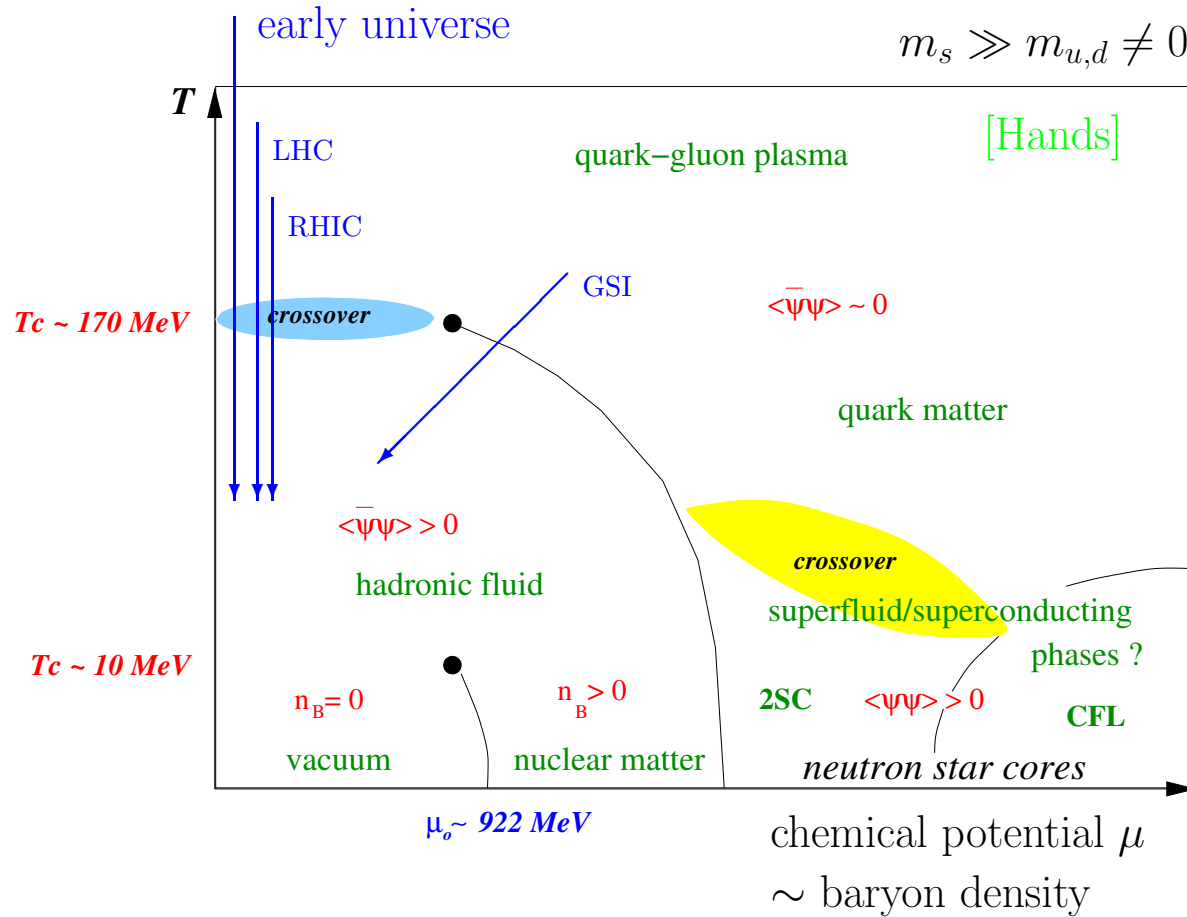
- Y.Aoki et al., JHEP 2006



quark masses seem to not matter too much – controlling/reducing UV effects important

### III. Phase diagram at $\mu \neq 0$

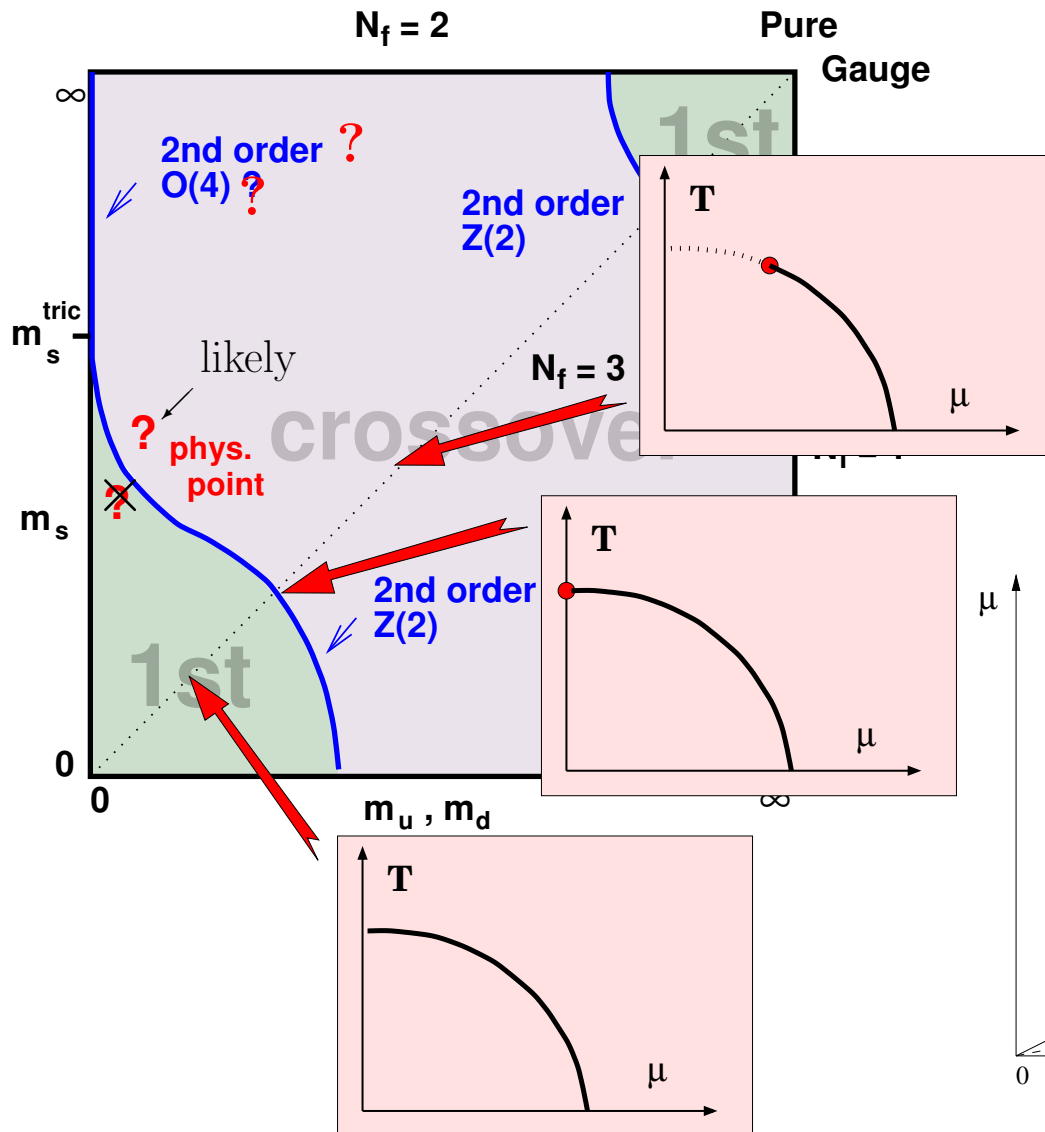
expected properties :



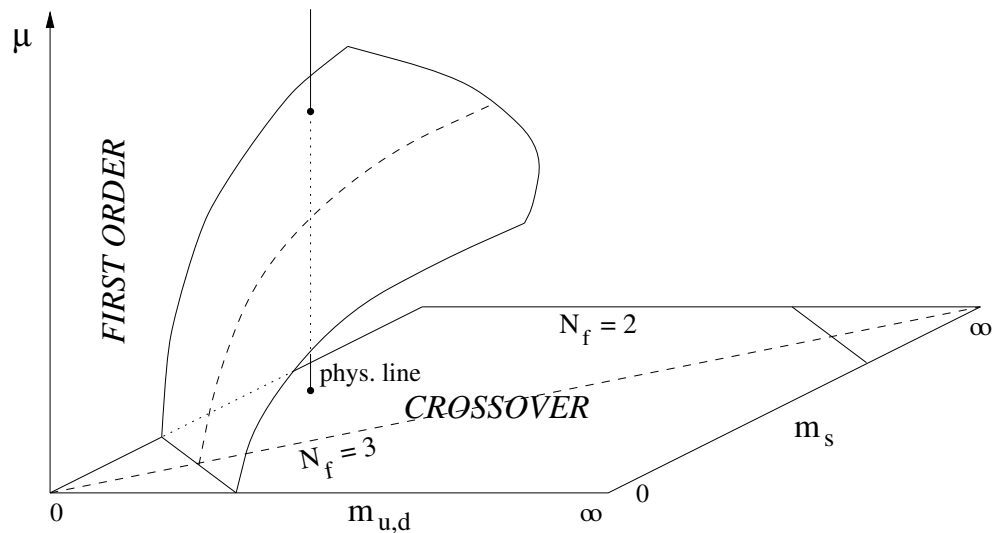
in detail dependent on  
masses of light flavors

$$\begin{aligned}
 m_{u,d} &\ll m_s & N_F &= 2 \\
 m_{u,d} &< m_s & N_F &= 2 + 1 \\
 m_{u,d} &\simeq m_s & N_F &= 3
 \end{aligned}$$

see e.g. Rajagopal, Wilczek, hep-ph/0011333



summarized:



the problem :

$$Z_{GC}(T, V, \mu) = \int \mathcal{D}U_\mu \mathcal{D}q \mathcal{D}\bar{q} \exp \{-S_G(U) + \bar{q}M(\mu)q\}$$

integrate over quark fields

$$Z_{GC}(T, V, \mu) = \int \mathcal{D}U_\mu \det M(\mu) \exp \{-S_G(U)\}$$

- for  $\mu \neq 0$  :  $\det M(\mu)$  complex  $\Rightarrow$  can not be used as statistical weight in Monte Carlo
- reformulate:  $\det M(\mu) = |\det M(\mu)| e^{i\Theta}$  and use phase  $\Theta$  as (part of the) observable:
 
$$\langle \mathcal{O} \rangle_{\det M} = \langle \mathcal{O} e^{i\Theta} \rangle_{|\det M|} / \langle e^{i\Theta} \rangle_{|\det M|}$$
- but :  $\langle e^{i\Theta} \rangle_{|\det M|} \sim e^{-V}$  ‘sign problem’

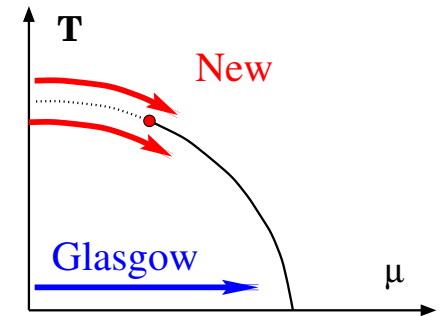
\* **‘Reweighting’**

[Glasgow; Fodor, Katz]

simulate at parameters  $p_0 = (g, m, \mu)_0$  and reweight to  $p = (g, m, \mu)$

$$\mathcal{D}U e^{-S_G(p)} \det M(p) = \underbrace{\mathcal{D}U e^{-S_G(p_0)} \det M(p_0)}_{\text{simulation}} \star \underbrace{e^{-[S_G(p) - S_G(p_0)]} \frac{\det M(p)}{\det M(p_0)}}_{\text{correction-factor}}$$

- limited by overlap



\* **‘Taylor-expansion’**

[Bielefeld-Swansea; Gavai, Gupta]

$$\langle \mathcal{O} \rangle \left( \frac{\mu}{T} \right) = \langle \mathcal{O} \rangle_{\mu=0} + \langle \tilde{\mathcal{O}}_2 \rangle_{\mu=0} \star \left( \frac{\mu}{T} \right)^2 + \langle \tilde{\mathcal{O}}_4 \rangle_{\mu=0} \star \left( \frac{\mu}{T} \right)^4 + \dots \quad \text{with} \quad \tilde{\mathcal{O}}_k = \frac{1}{k!} \frac{\partial^k \mathcal{O} \det M}{\partial \mu^k}$$

- limited by convergence radius

\* **‘imaginary  $\mu$ ’**

[Forcrand, Philipsen; D’Elia, Lombardo]

- $\mu = i\mu_I \Rightarrow \det M$  real and positive
- analytic continuation to real  $\mu$

- limited by  $Z(\mu_I/T) = Z(\mu_I/T + 2\pi/3)$

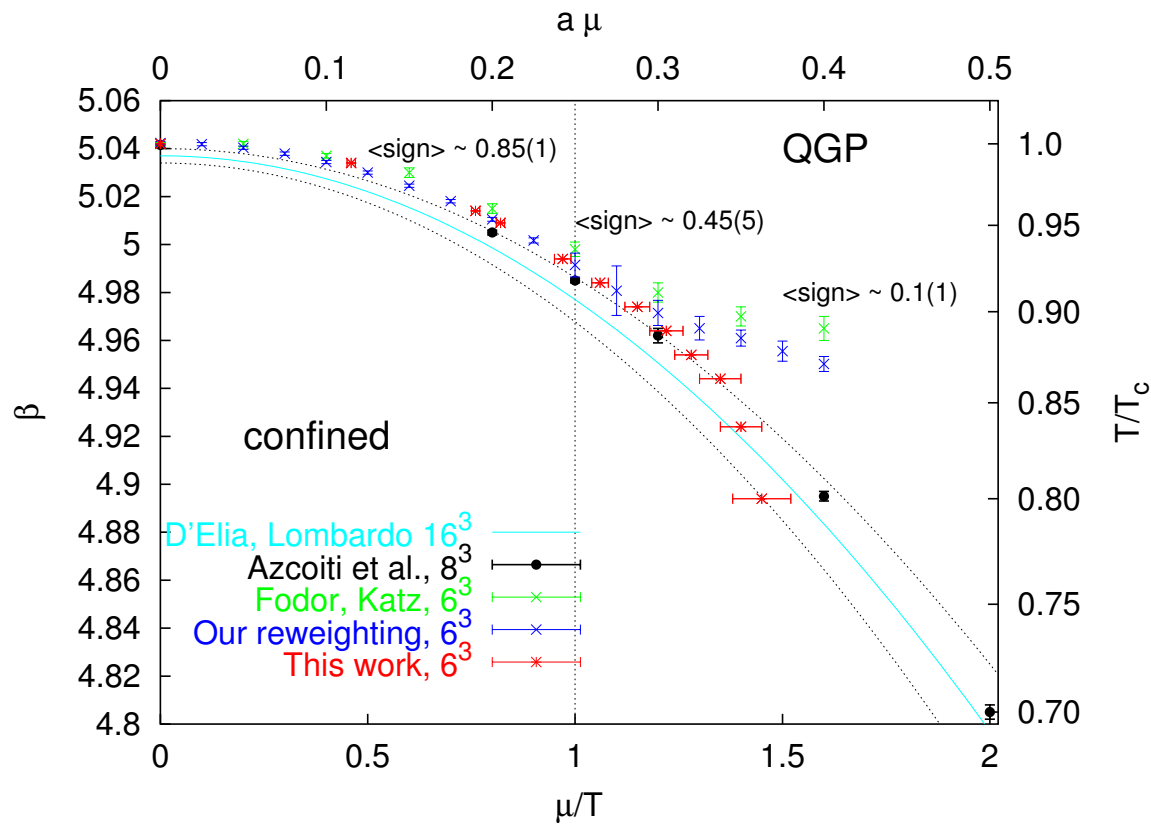
\* 'canonical'

[Forcrand,Kratochvila]

$$Z_C(B) = \frac{1}{2\pi} \int d\left(\frac{\mu_I}{T}\right) \exp\left\{-i3B\frac{\mu_I}{T}\right\} Z_{GC}(\mu = \mu_I)$$

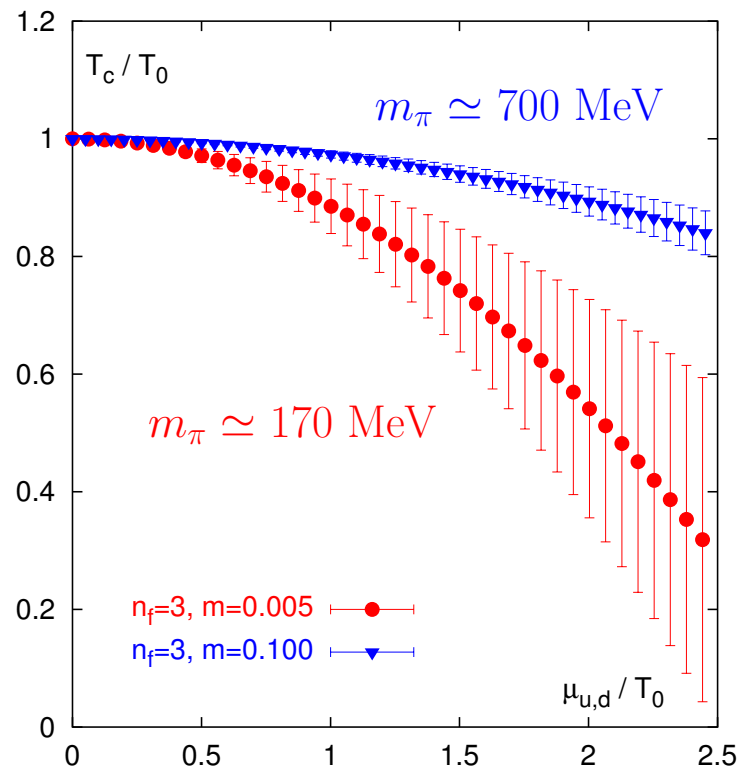
- sample at fixed  $\mu_I$
- Fourier transform each determinant  $\rightarrow$  work  $\sim N_\sigma^9 \times N_\tau$
- combine with reweighting in  $\mu_I$
- back to  $Z_{GC}(\mu)$  by  $\Sigma_B \exp\left\{+3B\frac{\mu}{T}\right\} Z_C(B)$

# Phase Diagram $T - \mu$ : comparing apples with apples



i) reweighting becomes unreliable

- applicable at small values for  $\mu$  in the phenomenologically relevant range for RHIC, LHC
- first, exploratory results in qualitative agreement, further systematic investigations required
- in particular at smaller quark masses :



$N_F = 3$ , improved action

$$\frac{T_c(\mu)}{T_c(0)} = 1 - 0.025(6) \left( \frac{\mu}{T_c(0)} \right)^2$$

$$\frac{T_c(\mu)}{T_c(0)} = 1 - 0.114(46) \left( \frac{\mu}{T_c(0)} \right)^2$$

(perturbative  $\beta$ -function  $d\beta_c/d \ln a$ )

- considerable quark mass dependence

## IV. Equation of state at $\mu \neq 0$

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### Observables

pressure ( $\mu = (\mu_u, \mu_d, \dots)$ )

$$\frac{p}{T^4} = \Omega(T, \mu) = \frac{1}{VT^3} \ln Z(T, \mu) = \sum_{n=0}^{\infty} c_n(T, m_q) \left(\frac{\mu}{T}\right)^n$$

with

$$c_n(T, m_q) = \frac{1}{n!} \frac{1}{VT^3} \left. \frac{\partial^n \ln Z(T, \mu)}{\partial (\mu/T)^n} \right|_{\mu=0}$$

number density

$$\frac{n_q}{T^3} = \frac{1}{VT^3} \frac{\partial \ln Z(T, \mu)}{\partial (\mu/T)} = \sum_{n=2}^{\infty} n c_n(T, m_q) \left(\frac{\mu}{T}\right)^{n-1}$$

interaction measure

$$\frac{\epsilon - 3p}{T^4} = \sum_{n=0}^{\infty} c'_n(T, m_q) \left(\frac{\mu}{T}\right)^n$$

with

$$c'_n(T, m_q) = T \frac{dc_n(T, m_q)}{dT}$$

from those, energy density

$$\frac{\epsilon}{T^4} = \sum_{n=0}^{\infty} (3c_n(T, m_q) + c'_n(T, m_q)) \left(\frac{\mu}{T}\right)^n$$

and entropy density

$$\frac{s}{T^3} = \frac{\epsilon + p - \mu n_q}{T^4} = \sum_{n=0}^{\infty} ((4 - n)c_n(T, m_q) + c'_n(T, m_q)) \left(\frac{\mu}{T}\right)^n$$

diagonal and off-diagonal susceptibilities

$$\frac{\chi_{ff}(T, \mu)}{T^2} = \frac{\partial^2 \Omega(T, \mu)}{\partial(\mu_f/T)^2}$$

$$\frac{\chi_{fk}(T, \mu)}{T^2} = \frac{\partial^2 \Omega(T, \mu)}{\partial(\mu_f/T)\partial(\mu_k/T)}$$

with  $\mu_q = \frac{1}{2}(\mu_u + \mu_d)$  and  $\mu_I = \frac{1}{2}(\mu_u - \mu_d)$

quark number susceptibility

$$\frac{\chi_q(T, \mu)}{T^2} = \frac{\partial^2 \Omega(T, \mu)}{\partial(\mu_q/T)^2} = 2(\chi_{uu} + \chi_{ud})$$

isovector susceptibility

$$\frac{\chi_I(T, \mu)}{T^2} = \frac{\partial^2 \Omega(T, \mu)}{\partial(\mu_I/T)^2} = 2(\chi_{uu} - \chi_{ud})$$

charge susceptibility

$$\frac{\chi_Q(T, \mu)}{T^2} = \frac{\partial^2 \Omega(T, \mu)}{\partial(\mu_Q/T)^2} = \frac{1}{9}(5\chi_{uu} - 4\chi_{ud})$$

measure fluctuations of quark number ( $q$ ), isospin ( $I$ ) and charge ( $Q$ )  $\rightarrow$  event-by-event fluctuations

## What is known analytically

(A) **high temperature** : perturbation theory

[Vuorinen]

$$\Omega(T, \mu) = \Omega^{(0)}(T, \mu) + g^2 \Omega^{(2)}(T, \mu) + g^3 \Omega^{(3)}(T, \mu) + \mathcal{O}(g^4)$$

with Stefan-Boltzmann (free gas) limit

$$\frac{p_{SB}}{T^4} = \Omega^{(0)}(T, \mu) = \frac{8\pi^2}{45} + \sum_{f=u,d,..} \left[ \frac{7\pi^2}{60} + \frac{1}{2} \left( \frac{\mu_f}{T} \right)^2 + \frac{1}{4\pi^2} \left( \frac{\mu_f}{T} \right)^4 \right]$$

diagonal suscept.

$$\frac{\chi_{ff}(T, \mu)}{T^2} = 1 + \frac{3}{\pi^2} \left( \frac{\mu_f}{T} \right)^2 + \mathcal{O}(g^2)$$

off-diagonal suscept.

$$\frac{\chi_{fk}(T, \mu)}{T^2} = g^3 \kappa \frac{\mu_f \mu_k}{T T} + \mathcal{O}(g^4)$$

$$\frac{\chi_{fk}(T, 0)}{T^2} = -\frac{5}{144\pi^6} g^6 \ln 1/g$$

(B) **low temperature** : hadron resonance gas model

$$\Omega_{HRG}(T, \mu_q, \mu_I) = \sum_{i \in \text{mesons}} \Omega_{m_i}^M(T, \mu_q, \mu_I) + \sum_{i \in \text{baryons}} \Omega_{m_i}^B(T, \mu_q, \mu_I)$$

where

$$\Omega_{m_i}^{M/B} = \frac{1}{2\pi^2} \left(\frac{m_i}{T}\right)^2 \sum_{\ell=1}^{\infty} \left\{ (-1)^{\ell+1} \right\} \ell^{-2} K_2\left(\frac{\ell m_i}{T}\right) z_i^\ell \quad \text{with } z_i = \exp((3B_i\mu_q + 2I_{3i}\mu_I)/T)$$

fugacities

- for baryons,  $\ell \geq 2$  terms can safely be neglected<sup>1</sup>  $\Rightarrow$  at  $\mu_I = 0$  :

$$\frac{p(T, \mu_q, \mu_I = 0)}{T^4} \simeq G(T) + F(T) \cosh\left(\frac{3\mu_q}{T}\right) \quad \Rightarrow \quad \frac{\chi_q}{T^2} = 9F(T) \cosh\left(\frac{3\mu_q}{T}\right)$$

likewise,  $\frac{\chi_I(T, \mu_q, \mu_I = 0)}{T^2} \simeq G^I(T) + F^I(T) \cosh\left(\frac{3\mu_q}{T}\right)$

- For all quantities  $X$  of the form  $X = G^X(T) + F^X(T) \cosh(3\mu_q/T)$  :

$$X = \sum_{n=0}^{\infty} c_n^X(T) (\mu_q/T)^n \quad \text{with} \quad \frac{c_{2n+2}^X}{c_{2n}^X} = \frac{9}{(2n+2)(2n+1)} \quad \text{for } n \geq 1$$

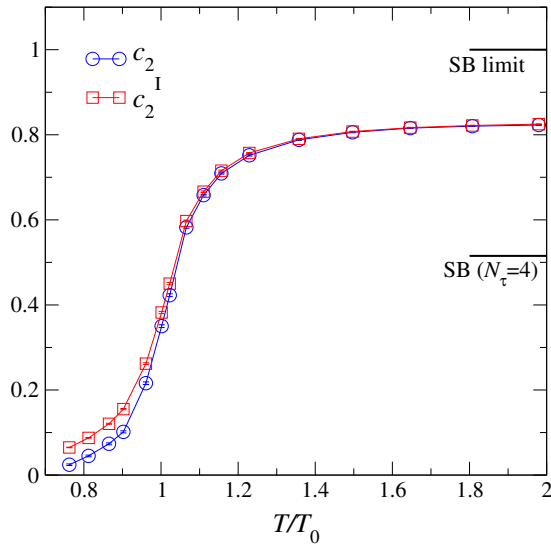
<sup>1</sup>  $K_2(x) \sim e^{-x}(1 + P(1/x))/\sqrt{x}$

pressure  $\frac{\Delta p}{T^4} = \frac{p(\mu_q)}{T^4} - \frac{p(\mu_q = 0)}{T^4} = c_2 \left(\frac{\mu_q}{T}\right)^2 + c_4 \left(\frac{\mu_q}{T}\right)^4 + c_6 \left(\frac{\mu_q}{T}\right)^6 + \dots$

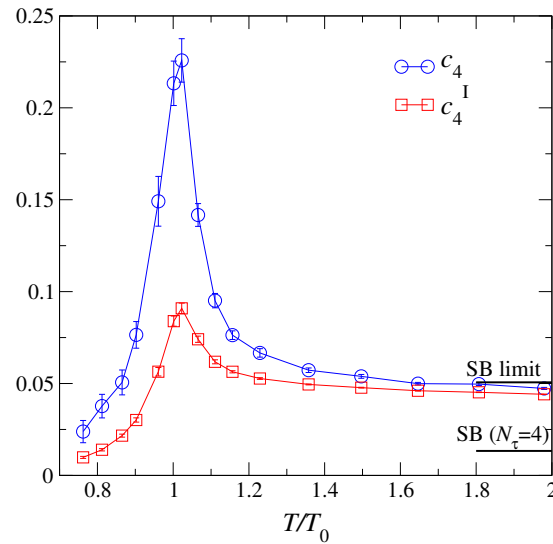
quark number  $\frac{n_q(T, \mu_q)}{T^3} = 2c_2 \left(\frac{\mu_q}{T}\right) + 4c_4 \left(\frac{\mu_q}{T}\right)^3 + 6c_6 \left(\frac{\mu_q}{T}\right)^5 + \dots$

q number suscept.  $\frac{\chi_q(T, \mu_q)}{T^2} = 2c_2 + 12c_4 \left(\frac{\mu_q}{T}\right)^2 + 30c_6 \left(\frac{\mu_q}{T}\right)^4 + \dots$

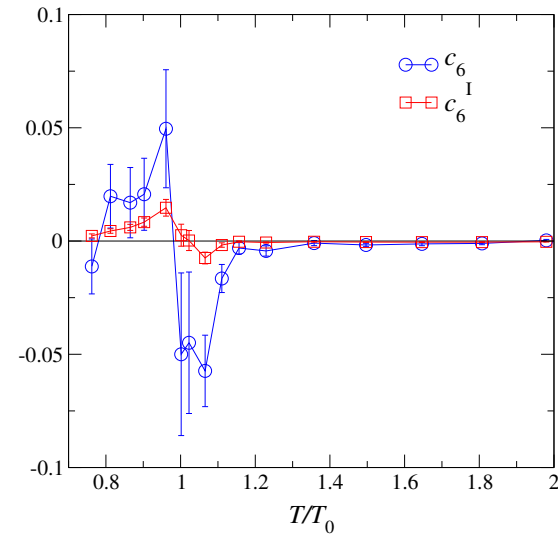
Isvector suscept.  $\frac{\chi_I(T, \mu_q)}{T^2} = 2c_2^I + 12c_4^I \left(\frac{\mu_q}{T}\right)^2 + 30c_6^I \left(\frac{\mu_q}{T}\right)^4 + \dots$



- approaching SB
- discretisation effects !

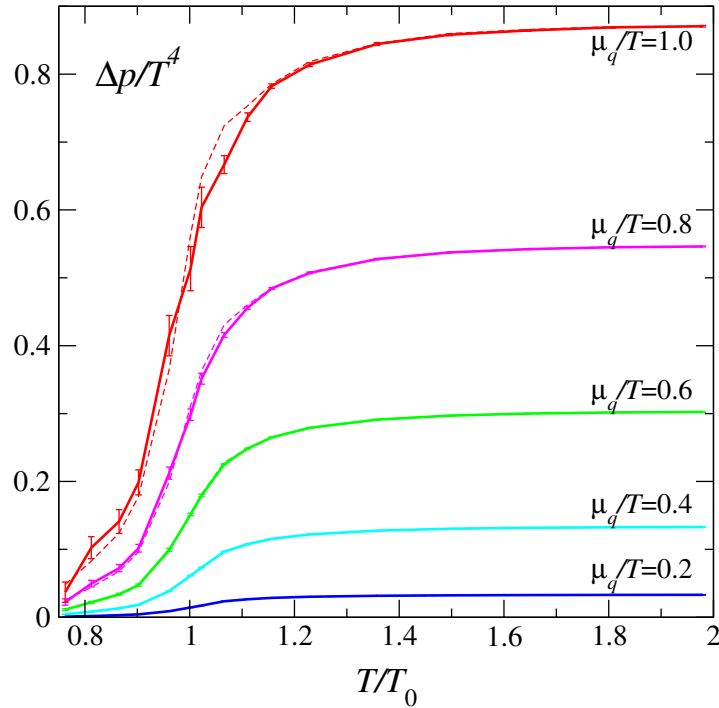


- peak around  $T_c$

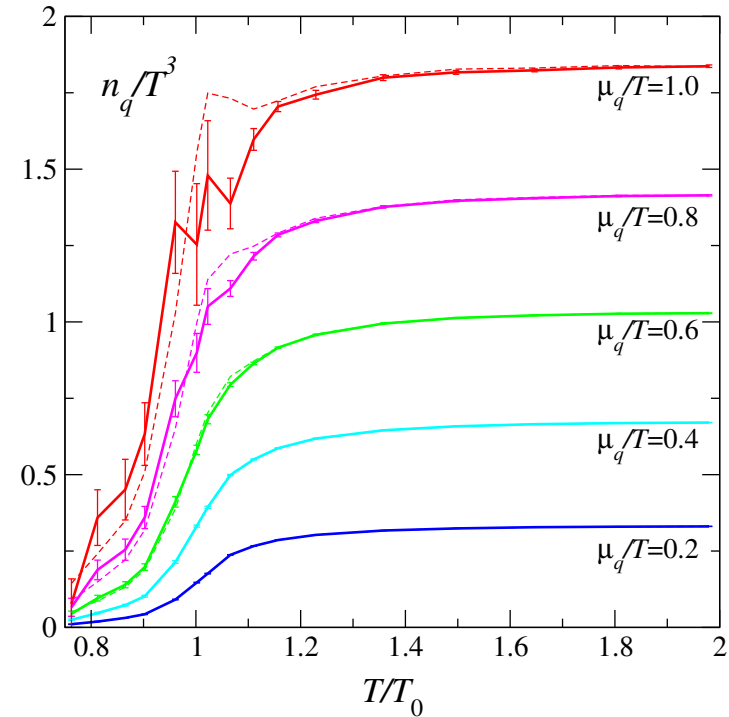


- sign change around  $T \lesssim T_c$
- small

pressure



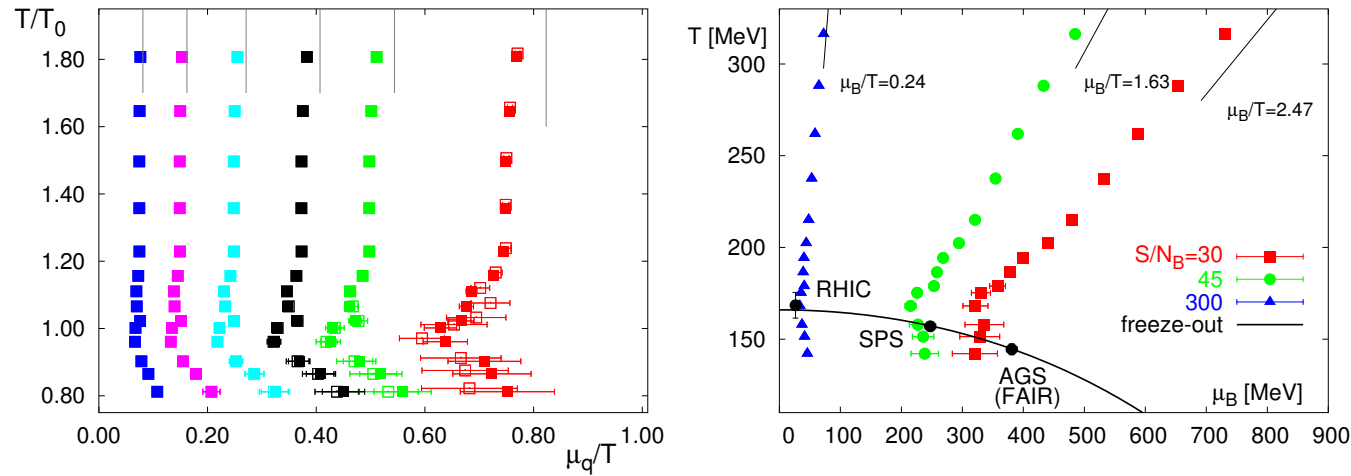
quark number density



- comparison: up to  $\mathcal{O}(\mu^4)$  (dashed) with up to  $\mathcal{O}(\mu^6)$  (full) suggests rapid convergence
- contribution to total p is small:  $p(\mu = 0)/T^4 \simeq \mathcal{O}(4)$

## Lines of fixed entropy over baryon number

it is generally believed that the fireball expansion follows a line of fixed  $S/N_B$



in the ideal gas limit

$$\frac{S}{N_B} = 3 \frac{\frac{37\pi^2}{45} + \left(\frac{\mu_q}{T}\right)^2}{\frac{\mu_q}{T} + \frac{1}{\pi^2} \left(\frac{\mu_q}{T}\right)^3} \quad \Rightarrow \quad \frac{\mu_q}{T} = \text{const (vertical lines)}$$

isentropic expansion lines for **SPS:  $S/N_B \simeq 45$**

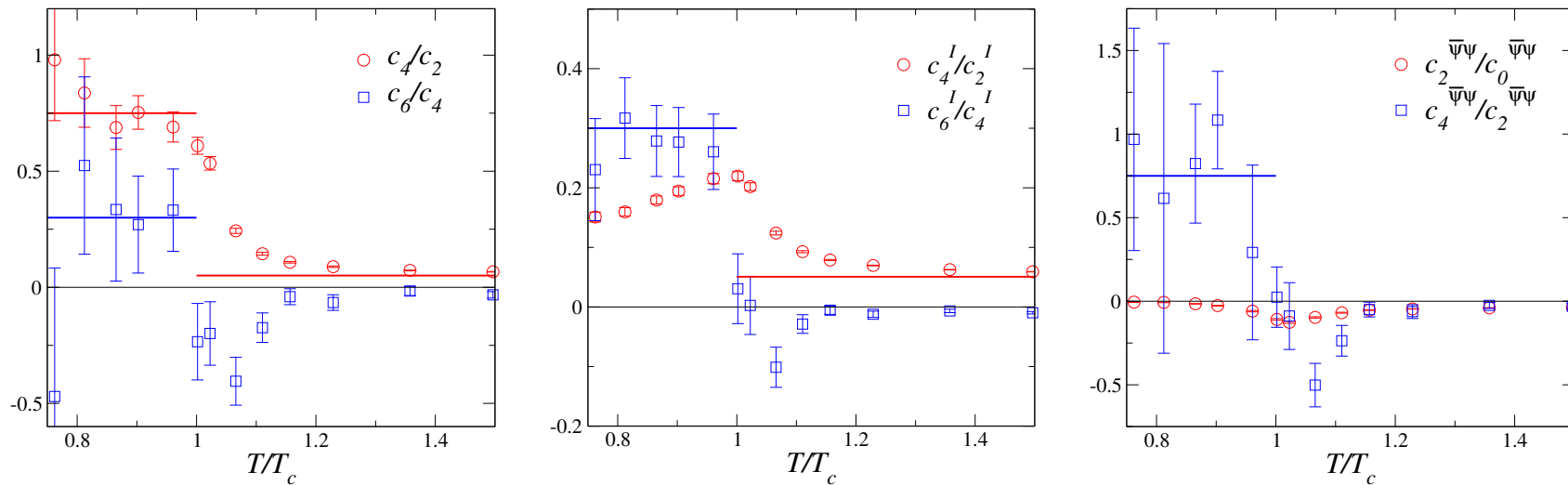
**RHIC:  $S/N_B \simeq 300$**

**FAIR:  $S/N_B \simeq 30$**

keep in mind: feasibility study of what one can do with lattice data

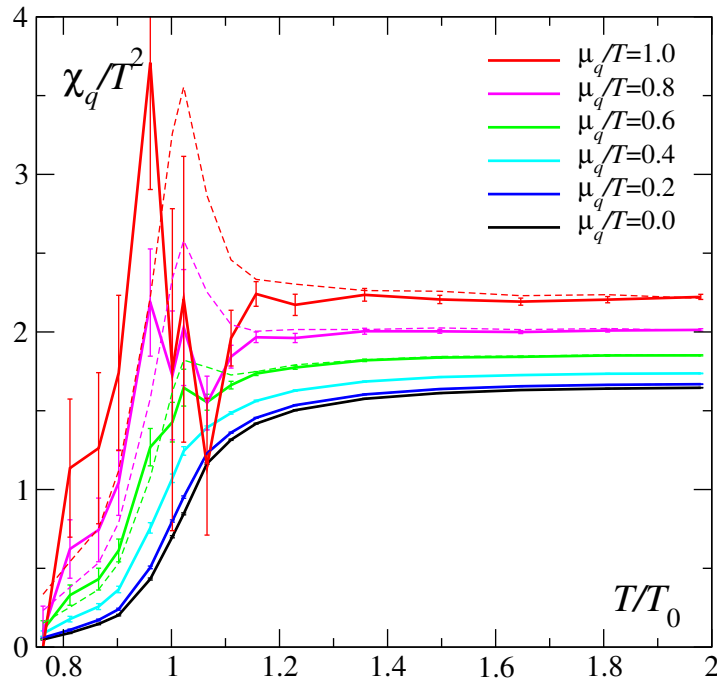
## Comparison with analytic results

recall: ratios  $\frac{c_{2n}}{c_{2n+2}}$  allow comparison with the hadron resonance gas model fairly detail independent

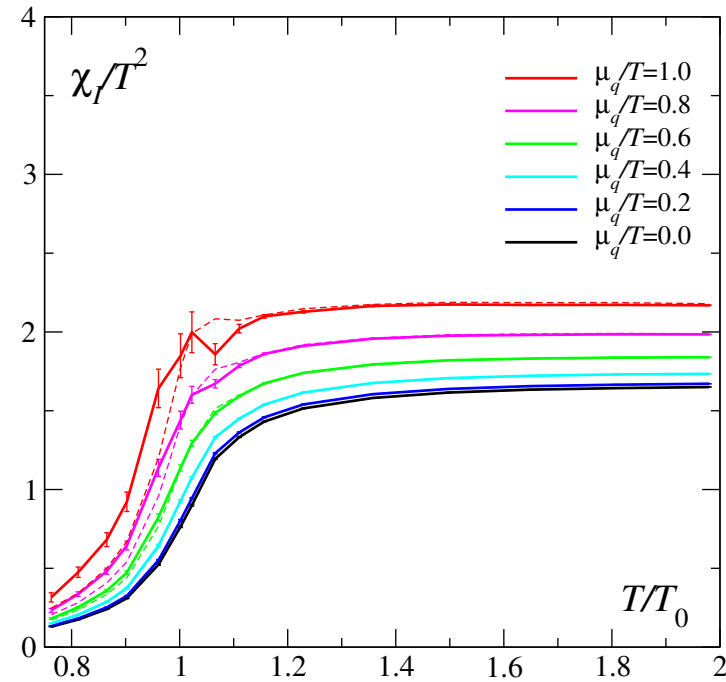


- above  $T_0$ : ratios approaching SB values
- below  $T_0$ : ratios except those involving  $c_0, c_2^I, c_0^{\bar{\psi}\psi}$  (depend on  $G^X(T)$ ) are
  - temperature independent
  - taking hadron resonance gas values  $\rightarrow$  do not indicate critical behavior

quark number susceptibility

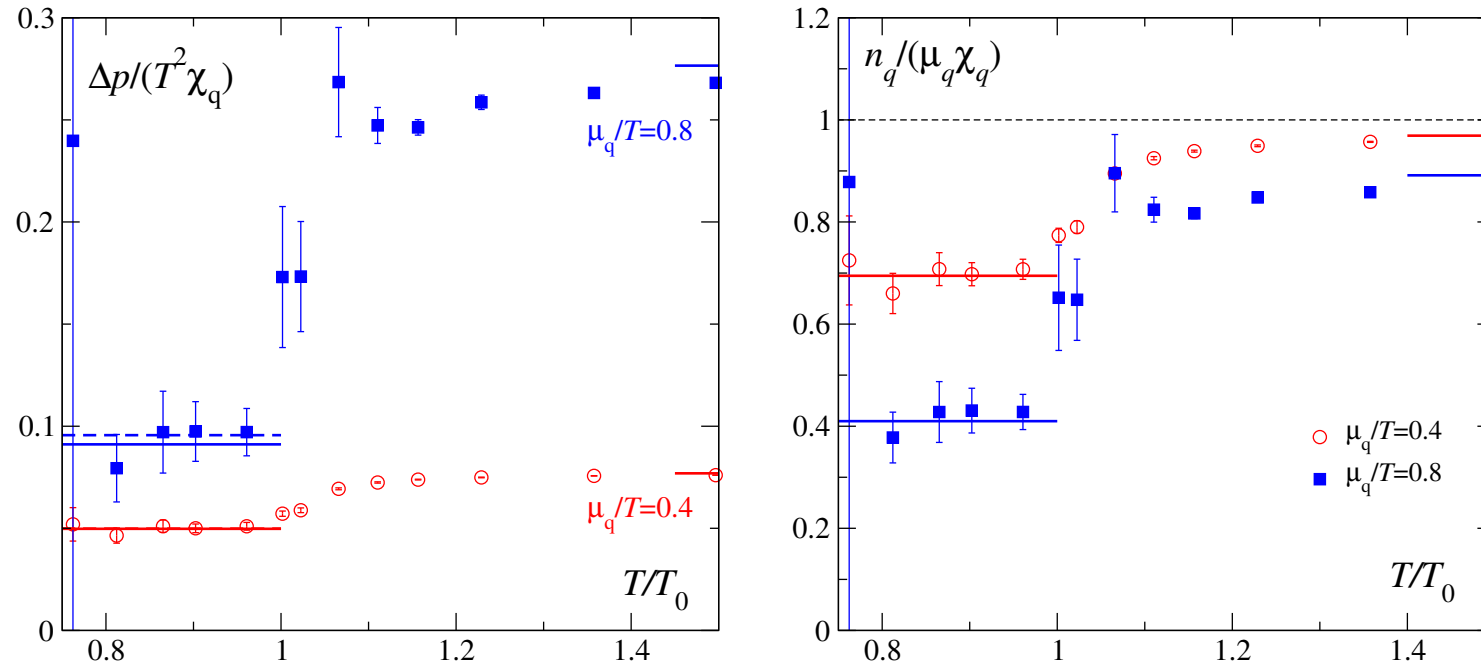


isovector susceptibility



- again comparison: up to  $\mathcal{O}(\mu^4)$  (dashed) with up to  $\mathcal{O}(\mu^6)$
- peak in  $\chi_q$  developing with increasing  $\mu$ , coming from  $c_4$
- $c_6$  shifts peak in  $\chi_q$  to smaller  $T$
- peak less convincing because of error bars and dip  $\rightarrow$  more statistics needed here
- **no** peak in  $\chi_I$   $\rightarrow$  strong correlations between  $\chi_{uu}$  and  $\chi_{ud}$

- $\chi_q$  rises rapidly with increasing  $\mu_q$ , but rise to a large extent due to rise in pressure



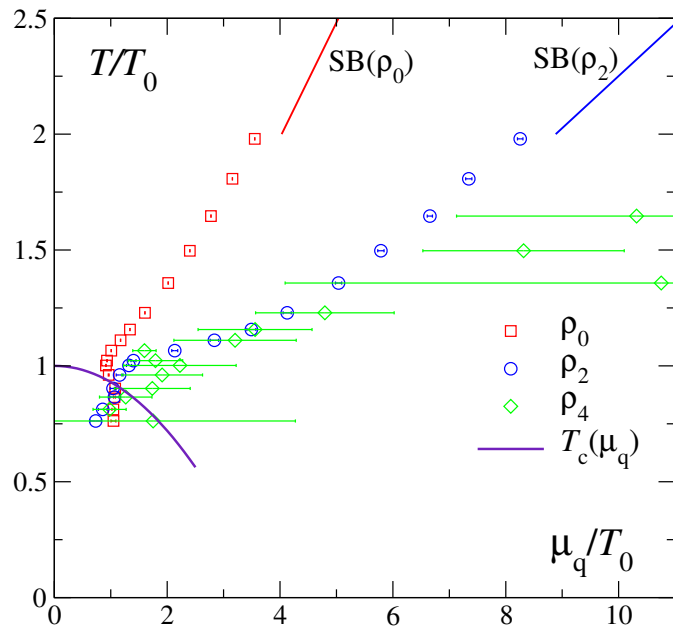
- $\frac{\partial p}{\partial n_q} = \frac{\partial p / \partial \mu_q}{\partial n_q / \partial \mu_q} = \frac{n_q}{\chi_q} = \frac{1}{\kappa_T n_q} \rightarrow 0$  at 2<sup>nd</sup> order phase transition (isothermal compressibility  $\kappa_T \rightarrow \infty$ )

- no indication of criticality

- but, for  $T \leq 0.96T_c$ , consistency with hadron resonance gas model (at  $\mu_I = 0$ )

$$\frac{n_q^{HRG}}{\mu_q \chi_q^{HRG}} = \frac{T}{3\mu_q} \tanh\left(\frac{3\mu_q}{T}\right)$$

### convergence radius



nearest (complex) singularity determines convergence radius

$$\rho = \lim_{k \rightarrow \infty} \rho_k = \lim_{k \rightarrow \infty} \sqrt{\frac{c_k}{c_{k+2}}}$$

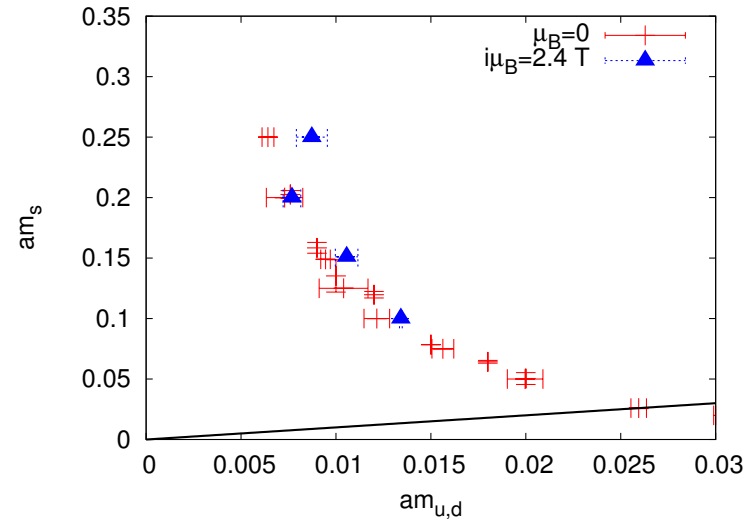
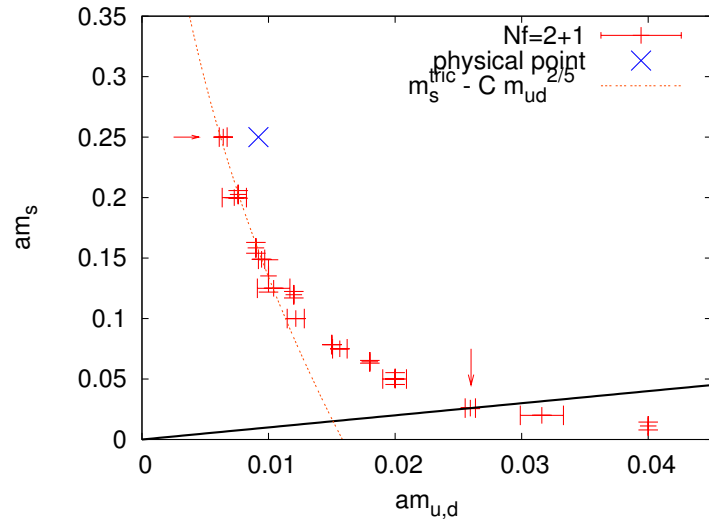
- SB limit:  $\rho_k = \infty$  for  $k \geq 4$
- for T big: approaching SB limit
- at  $T_c(\mu)$  :  $\rho_k \simeq 1$
- $c_k > 0 \Rightarrow$  convergence radius indicates critical point

Results:

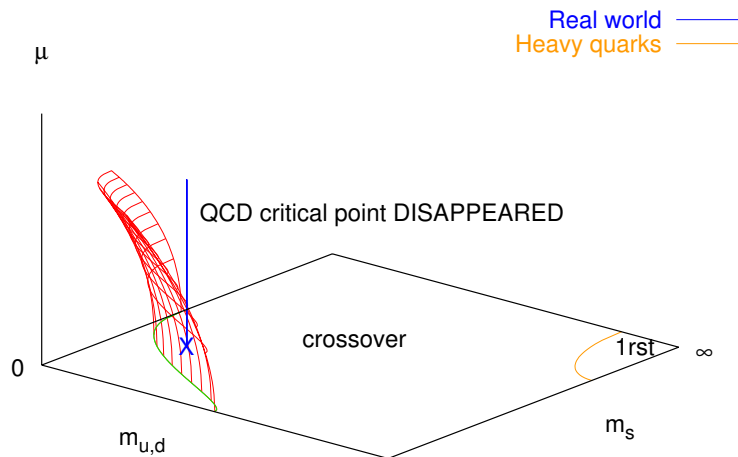
- Gavai, Gupta:  $\mu_{B,E} \simeq 180\text{MeV}$  (Taylor expansion)
- Fodor, Katz:  $\mu_{B,E} \simeq 360\text{MeV}$  (Lee-Yang zeroes)
- Bi-Swansea: LGT consistent with HRG, HRG analytic (Taylor expansion)

existence of a critical endpoint ?

[deForcrand, Philipsen]



critical region has the tendency to grow with  $\mu_I \Rightarrow$  shrink with real  $\mu$



but: finer lattices/improved actions needed  
 $N_\tau = 4$  here

## Outlook

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all these questions are being re-addressed

- at smaller lattice spacings
  - at smaller quark masses
- on TeraFlops machines