Calorons as a Challenge in Lattice QCD

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$\underline{SU(2)}$

Harrington & Shepard,
Phys. Rev. $\underline{D17}$ (1978) 2122.

Periodic (in time) array of (charge 1) instantons: Using 't Hooft ansatz

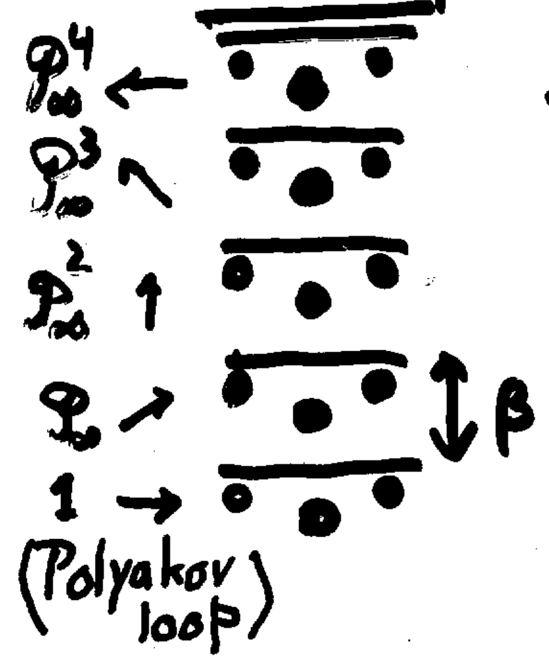
$$A_\mu(x) = \frac{i}{2}\bar{\eta}^a_{\mu\nu}\tau_a\partial_\nu \ln\phi(x) \qquad \frac{\Box\phi}{\phi}=0$$

$$\phi(x) = 1 + \sum_{n\in\mathbb{Z}} \frac{\rho^2}{(\vec{x}-\vec{a})^2+(t-a_0-n\beta)^2}$$

$$\underset{\substack{(\beta=1)\\(a_\mu=0)}}{=} 1+\frac{\pi\rho^2}{r}\,\frac{\sinh(2\pi r)}{\cosh(2\pi r)-\cos(2\pi t)} \qquad r=|\vec{x}|$$

For large ρ this approaches a BPS monopole (in singular gauge)
P. Rossi, Nucl. Phys. B149 (1979) 170

$\underline{\text{Overlap}}$: For large ρ scale is set by β (= distance between periodic copies)



What happens when periodic copies are relative gauge rotated?

$\beta \equiv \rho = 1$

$\pi \rho^2 = |\vec{y}_1 - \vec{y}_2| \beta$

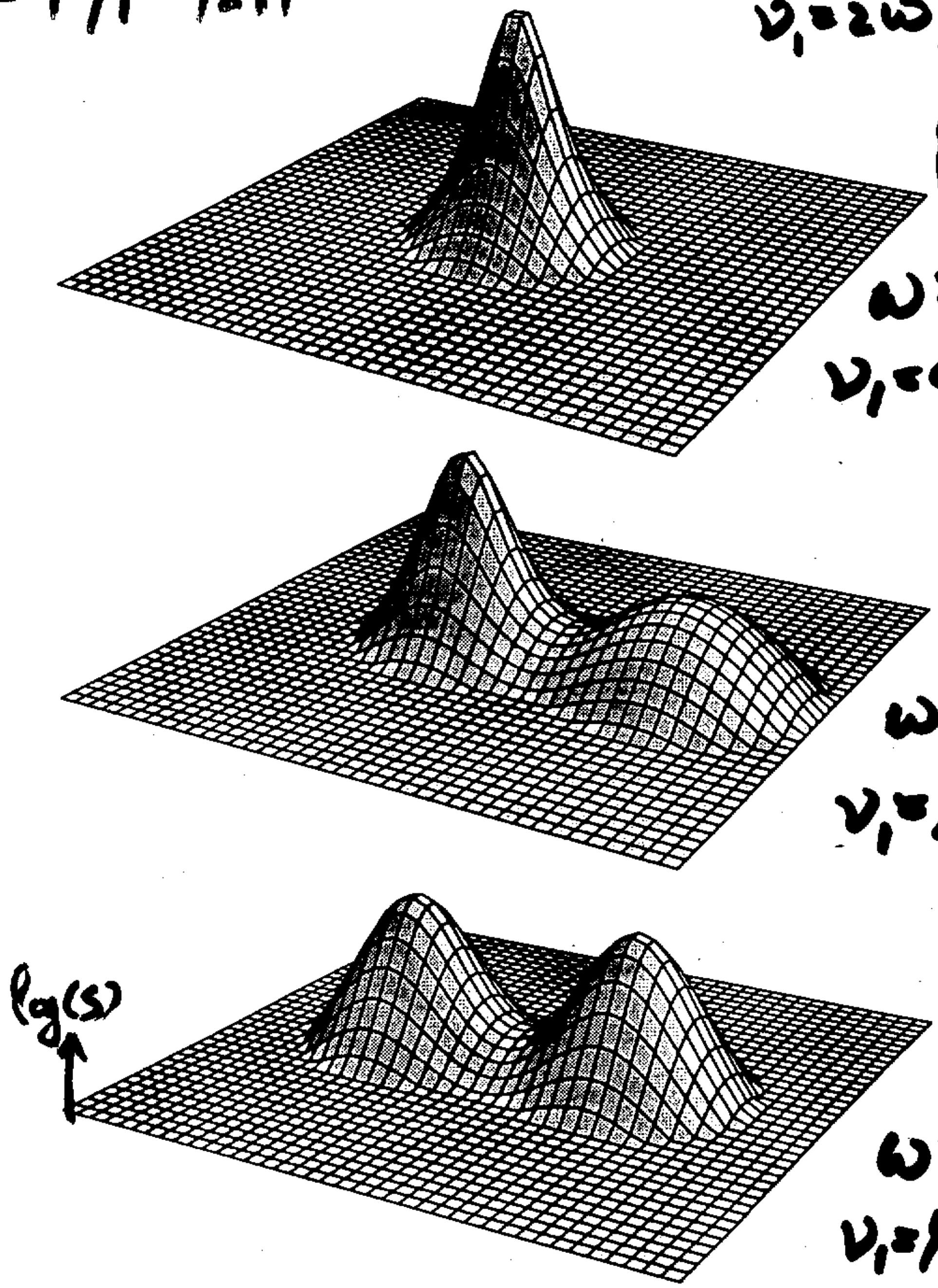
$P_\infty = \exp(2\pi i \vec{\omega} \cdot \vec{\tau})$

$\omega \equiv |\vec{\omega}|$

$\nu_1 = 2\omega, \nu_2 = 1 - 2\omega = 2\bar{\omega}$

($\nu_i \triangleq$ action fraction)

SU(2)



$\omega = 0$

$\nu_1 = 0, \nu_2 = 1$

$\omega = 1/8$

$\nu_1 = 1/4, \nu_2 = 3/4$

$\omega = 1/4$

$\nu_1 = 1/2, \nu_2 = 1/2$

Figure 1: Profiles for calorons at $\omega = 0$, 0.125, 0.25 (from top to bottom) with $\rho = 1$. The axis connecting the lumps, separated by a distance π (for $\omega \neq 0$), corresponds to the direction of $\hat{\omega}$. The other direction indicates the distance to this axis, making use of the axial symmetry of the solutions. Vertically is plotted the action density, at the time of its maximal value, on equal logarithmic scales for the three profiles. The profiles were cut off at an action density below $1/e$. The mass ratio of the two lumps is approximately $\omega/\bar{\omega}$, i.e. zero (no second lump), a third and one (equal masses), for the respective values of ω.

Density profiles from:

for $\underline{\underline{\text{all}}}$ ω $\boxed{\mathrm{tr}\, F_{\mu\nu}^2(x) = \Box^2 \ln \psi(x)}$

Interpretation of <u>solutions</u>:

Two constituent BPS monopoles with opposite magnetic charges with masses $8\pi^2 \nu_1/T$ and $8\pi^2\nu_2/T$ separated by a distance $\pi\rho^2/T$ along $\underline{\hat{\omega}}$

Parameters: (Moduli) $\boxed{8}$ $\begin{cases} \zeta = \zeta_\mu \sigma_\mu = \rho g \begin{cases} \rho = \text{scale} \\ g = \text{gauge} \end{cases} \\ a_\mu \end{cases}$ a_μ = position center-mass

For each $\vec{\omega}$ a different moduli <u>space</u>

$$\mathcal{M} = \underbrace{\mathbb{R}^3 \times S^1}_{a_\mu} \times \underbrace{\text{Taub-NUT}}_{\zeta} / \underbrace{Z_2}_{\zeta \leftrightarrow -\zeta}$$

$M / 16$

Also understood from Nahm transformation for monopoles: W. Nahm, Lect. Notes. Phys. 201 ('84) K. Lee, hep-th/9802012; K. Lee & C. Lu hep-th/9802108.

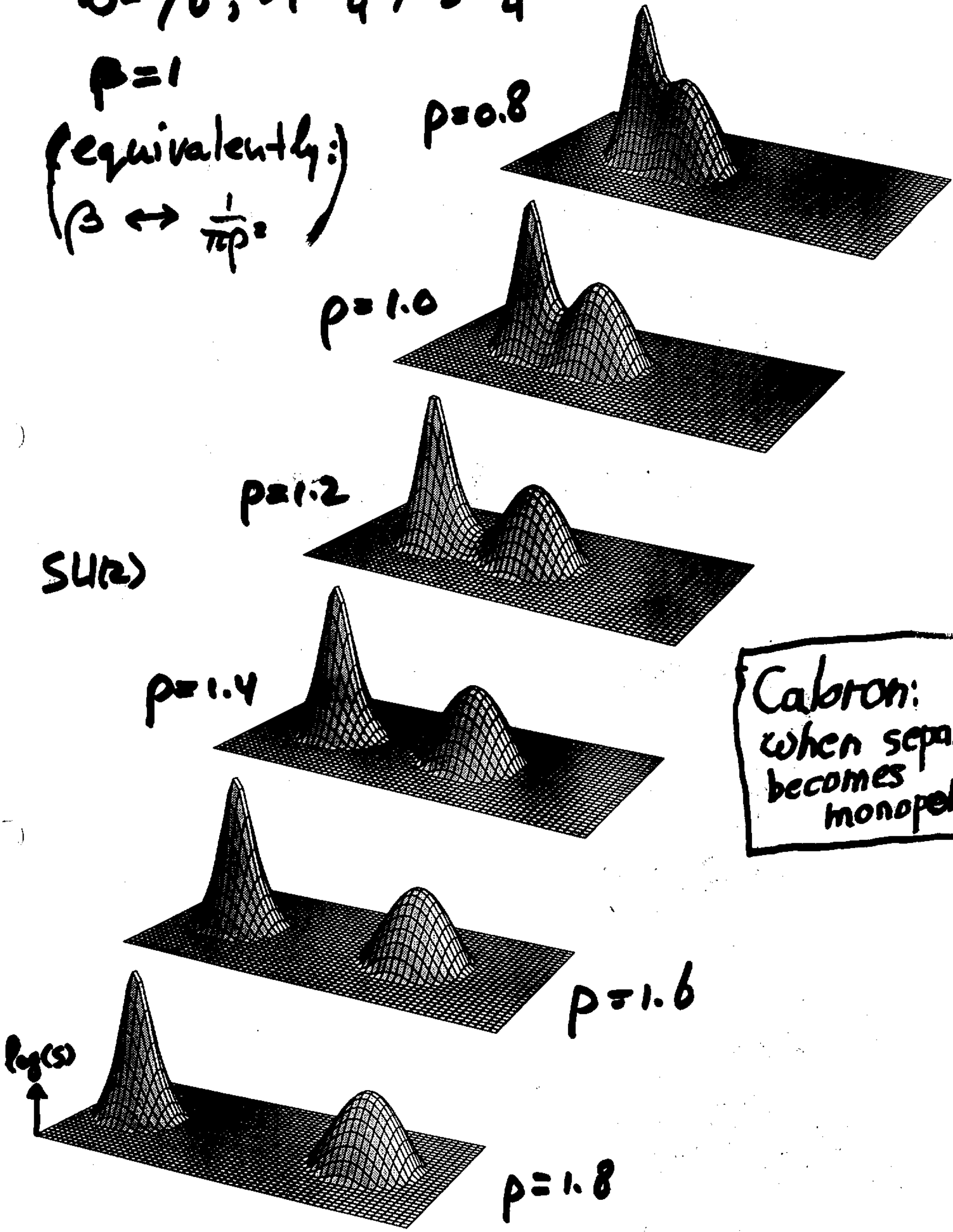
$\mathcal{P}_\infty = \exp(2\pi i \omega \tau_3)$

$\omega = 1/8$, $\nu_1 = \frac{1}{4}$, $\nu_2 = \frac{3}{4}$

$\beta = 1$

(equivalently: $\beta \leftrightarrow \frac{1}{\pi\rho^2}$)

SU(2)



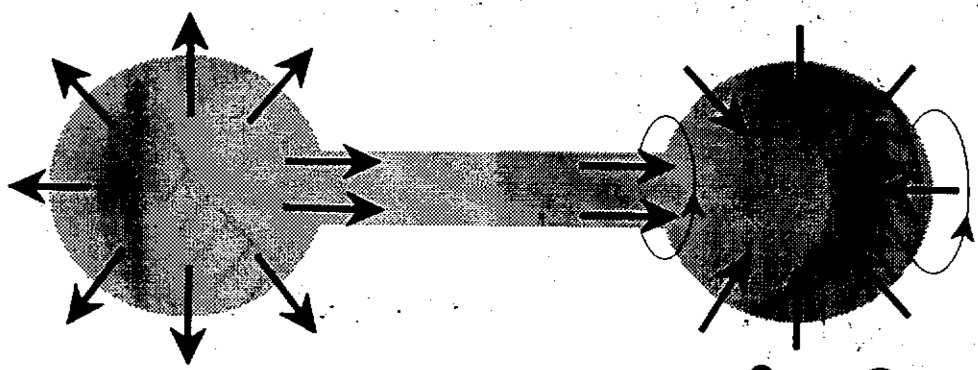
Calorn: when separ. becomes monopoles

$\omega = 1/8$; cutoff $1/e^2$

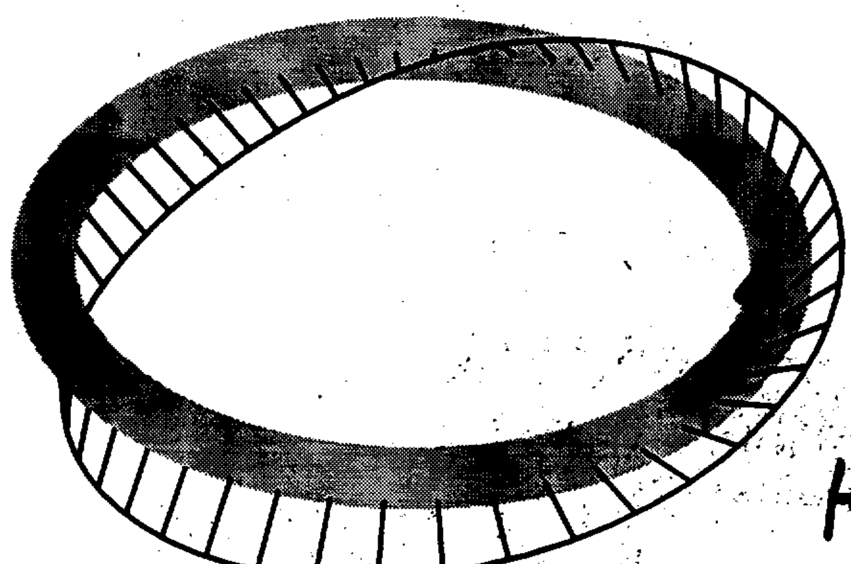
$\rho = 0.8, 1, 1.2, 1.4, 1.6, 1.8$

C. Taubes ('82)

See: Cargèse '83 - Plenum '84, p. 563



$S^2 = SU(2)/U(1)$



Hopf map winding # 1

Monopole loop S^1

"Frame" $SU(2)/U(1) = S^2$

$\Rightarrow S^1 \times_{\text{twisted}} S^2 = S^3$

Made more precise by O. Jahn*

*) O. Jahn, J. Phys. A33 (2000) 2997

T.C. Kraan & P.v.B., PLB 428 ('98) 268, NPB 533 ('98) 637.

SU(2) $\underline{P_\infty = \exp(2\pi i \vec{\omega}\cdot\vec{\tau}) \neq 1}$ $\omega \in [0, \tfrac{1}{2}]$

We may rotate to $P_\infty = \exp(2\pi i \omega \tau_3)$

Periodic (in time) array of (charge 1) instantons, "twisted" in colorspace by P_∞

$$A_\mu(x) = \frac{i}{2}\bar{\eta}^3_{\mu\nu}\tau_3\partial_\nu \ln\phi(x) \qquad \beta = 1$$

$$+\frac{i}{2}\phi(x)\,\mathrm{Re}\left\{(\tau_1 + i\tau_2)(\bar{\eta}^1_{\mu\nu} - i\bar{\eta}^2_{\mu\nu})\partial_\nu \chi(x)\right\}$$

$$\phi(x) = \psi(x)/\hat{\psi}(x) \qquad \boxed{\mathrm{tr}\,F^2_{\mu\nu}(x) = \Box^2 \log\psi(x)}$$

$$\chi(x) = \frac{\pi\rho^2 e^{4\pi i\omega t}}{\psi(x)}\left(\frac{s_1}{r_1}e^{-2\pi i t} + \frac{s_2}{r_2}\right)$$

$$\hat{\psi}(x) = c_1 c_2 + \frac{r_1^2 + r_2^2 - \pi^2\rho^4}{2r_1 r_2}s_1 s_2 - \cos(2\pi t)$$

$$\psi(x) = \hat{\psi}(x) + \pi\rho^2\left(\frac{s_1 c_2}{r_1} + \frac{s_2 c_1}{r_2} + \frac{\pi\rho^2 s_1 s_2}{r_1 r_2}\right)$$

$$s_i \equiv \sinh(2\pi\nu_i r_i)\,;\; c_i \equiv \cosh(2\pi\nu_i r_i)$$

$$\nu_1 \equiv 2\omega\,;\; \nu_2 \equiv 1 - 2\omega \equiv 2\bar{\omega}$$

$$r_1 = |\vec{x} - \pi\rho^2\nu_2\hat{\omega}|\,;\; r_2 = |\vec{x} + \pi\rho^2\nu_1\hat{\omega}|$$

$$\omega \to 0:\; \nu_1 = 0,\, \nu_2 = 1,\, \hat{\psi} = c_2 - \cos(2\pi t),\, \psi = \hat{\psi} + \frac{\pi\rho^2 s_2}{r_2}$$

$$\chi = 1 - \phi^{-1}$$

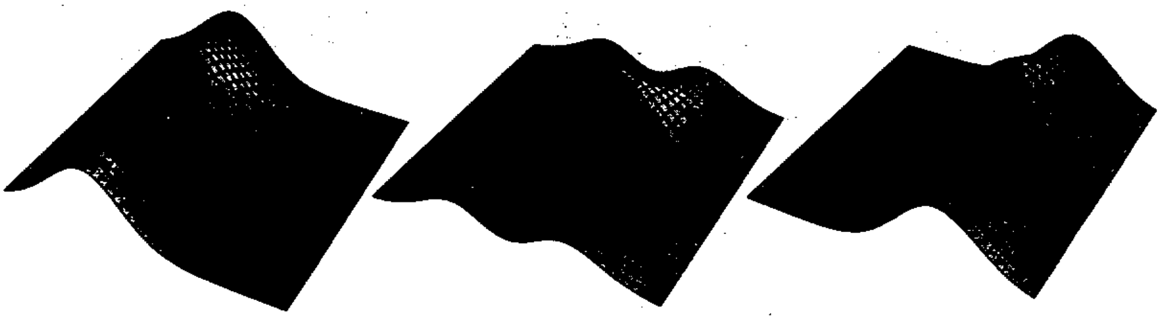


Figure 7: Fermion zero-mode densities as a function of t and z for a charge 1 caloron ($\mu_2 = \frac{1}{4}$ and $\rho = \frac{1}{2}$) with periodic (right) and anti-periodic (left) boundary conditions, compared to the action density (middle).

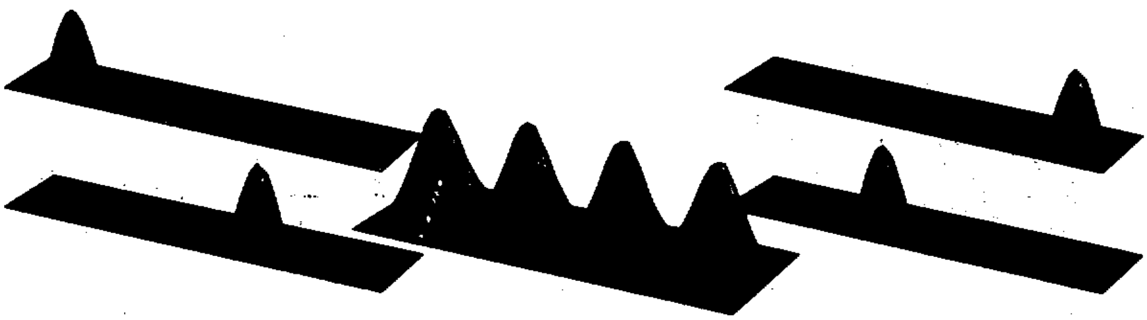


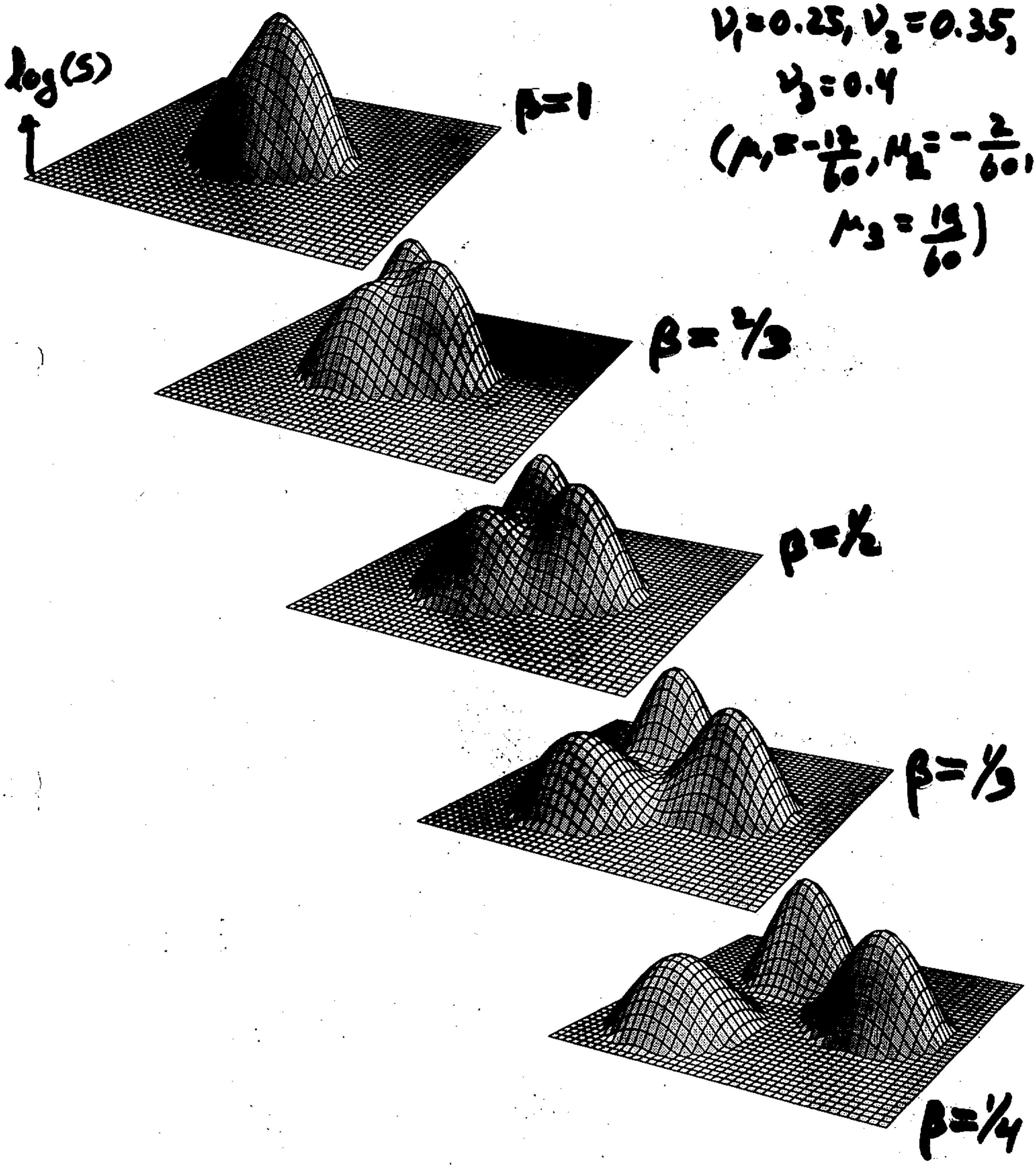
Figure 3: Zero-mode densities for a typical charge 2, $SU(2)$ axially symmetric solution. For comparison the action density (cmp. Fig. 2 of Ref. [11]) is shown in the middle. All are on a logarithmic scale, cutoff below e^{-3}. On the left is shown the two periodic zero-modes ($z = 0$) and on the right the two anti-periodic zero-modes ($z = 1/2$).

Figure 3: The action density in the plane of the constituents at $t = 0$ and the densitities for the two zero-modes, using either periodic (left) or anti-periodic (right) boundary conditions for an SU(2) charge 2 caloron in the so-called "crossed" configuration with $\mathbf{k} = 0.997$, $D = 8.753$ and $\mathrm{tr}\,\mathcal{P}_\infty = 0$.

Figure 4: As above, but for $\mathbf{k} = 0.962$ and $D = 3.894$. (non-static)

SU(3)

$\nu_1 = 0.25,\ \nu_2 = 0.35,$
$\nu_3 = 0.4$
$(\mu_1 = -\frac{17}{60},\ \mu_2 = -\frac{2}{60},$
$\mu_3 = \frac{19}{60})$



SU(3)

$SU(n)$

$\beta = 1$, top. ch. $= 1$

$$\mathrm{tr}\, F_{\mu\nu}^2(x) = \partial_\mu^2 \partial_\nu^2 \log \psi(x)$$

$$\psi(x) = \frac{1}{2}\mathrm{tr}\left(A_n A_{n-1} \cdots A_1\right) - \cos(2\pi x_0)$$

$$A_m = \begin{pmatrix} r_m & |\vec{y}_m - \vec{y}_{m+1}| \\ 0 & r_{m+1} \end{pmatrix} \begin{pmatrix} c_m & s_m \\ s_m & c_m \end{pmatrix} \frac{1}{r_m}$$

$$r_m = |\vec{x} - \vec{y}_m|$$

$$c_m = \cosh(2\pi \boxed{\nu_m} r_m)$$

$$s_m = \sinh(2\pi \boxed{\nu_m} r_m)$$

$SU(2)$:
$|\vec{y}_1 - \vec{y}_2| = \pi \rho^2$

$\mu_1 = -\omega,\ \mu_2 = \omega$

$$P(\vec{x}) = P\exp \int_0^\beta dx_0\, A_0(\vec{x}, x_0)$$

$$\xrightarrow[|\vec{x}| \to \infty]{} P_\infty = \exp\left[2\pi i\, \mathrm{diag}(\mu_1 \cdots \mu_n)\right]$$

$$\sum \mu_i = 0 \quad \mu_1 \leq \mu_2 \cdots \leq \mu_n \leq \mu_{n+1} \equiv \mu_1 + 1$$

$$\nu_m \equiv \mu_{m+1} - \mu_m \longrightarrow \text{Mass} = 8\pi^2 \nu_m / \beta$$

$$\boxed{\sum_m \nu_m = 1 \longleftrightarrow S = 8\pi^2}$$

$\beta = 1,\ \rho = 1.2$

$\mu_1 = -\omega = -\mu_2 = -1/8$

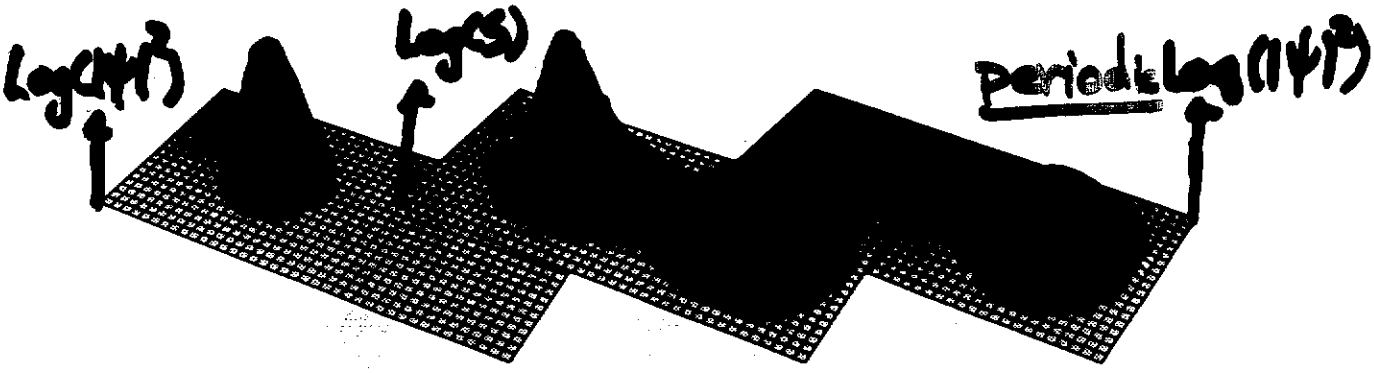


Figure 1: For the two figures on the sides we plot on the same scale the logarithm of the zero-mode densities (cutoff below $1/e^5$) for $\omega = 1/8$ (left Ψ^- / right Ψ^+) and $\omega = 3/8$ (right Ψ^- / left Ψ^+), with $\beta = 1$ and $\rho = 1.2$. In the middle figure we show for the same parameters (both choices of ω give the same action density) the logarithm of the action density (cutoff below $1/2e^2$).

$\nu_1 = 2\omega,\ \nu_2 = 1 - 2\omega$

$\mu_1 = -\omega,\ \mu_2 = \omega$

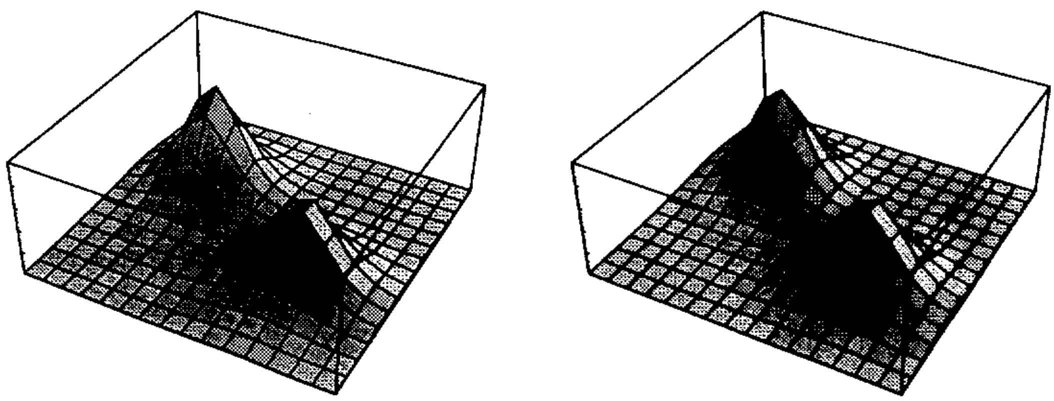


Figure 3: Zero-mode density profiles for the two zero-modes of the lattice caloron (left) on a 4×16^3 lattice for $\vec{k} = (1,1,1)$, created with improved cooling ($\varepsilon = 0$). The profiles fit well to the two zero-modes for the infinite volume analytic caloron solution (shown on the right at $y = t = 0$) with $\omega = \frac{1}{4}$ and constituents at $\vec{y}_1 = (2.50, 0.12, 0.95)$ and $\vec{y}_2 = (1.38, -0.24, 2.67)$, in units where $\beta = l_t = 1$ (or $a = \frac{1}{4}$) and the left most lattice point corresponding to $x = z = 0$. The plots give the added densities of the two zero-modes.

(with M. García Pérez, A. González-Arroyo and C. Pena)

hep-th/9905016/9905138

Fermion zero-mode $\quad SU(n)$

$$D\Psi = \bar\sigma_\mu(\partial_\mu + A_\mu)\Psi$$

$\boxed{\beta=1}$

$$\bar\sigma_\mu = (1_2; -i\vec\tau) \quad \Psi(t+\beta,\vec x) = e^{2\pi i z}\Psi(t;\vec x)$$

$z=\frac{1}{2}$ anti-periodic | $z=0$ periodic

$$|\Psi(x)|^2 = -\frac{1}{(2\pi)^2}\partial_\mu^2 \hat f_x(z,z)$$

for $z\in[\mu_m,\mu_{m+1}]$

$$\hat f_x(z,z) = \frac{\pi\langle v_m(z)|A_{m-1}\cdots A_1 A_n\cdots A_1|w_m(z)\rangle}{r_m\,\psi(x)}$$

$$V_m^1(z) = -W_m^2(z) = \sinh(2\pi(z-\mu_m)r_m)$$
$$V_m^2(z) = W_m^1(z) = \cosh(2\pi(z-\mu_m)r_m)$$

When $|\vec y_m - \vec y_{m-1}| \to \infty$ for all m

$$\hat f_x(z,z) \to \frac{\sinh(2\pi(z-\mu_m)r_m)\sinh(2\pi(z-\mu_{m+1})r_m)}{(2\pi)^{-1}r_m\sinh(2\pi\nu_m r_m)}$$

$\nu_m \to \mu_{m+1}-\mu_m$

For $SU(2)$ this gives:

$$\hat f_x(z,z) \to \frac{\pi\tanh(\pi\nu_m r_m)}{r_m}$$

$z=0 \to m=1 \;;\; z=\frac{1}{2} \to m=2$

For SUSY-YM each constituent monopole carries 2 gluino zero-modes. (adjoint!)
This saturates $\langle\lambda\lambda\rangle$ and resolves the an old problem in computing the Witten index (N.M. Davies, T.J. Hollowood, V.V. Khoze, M.P. Mattis, hep-th/9905015.)

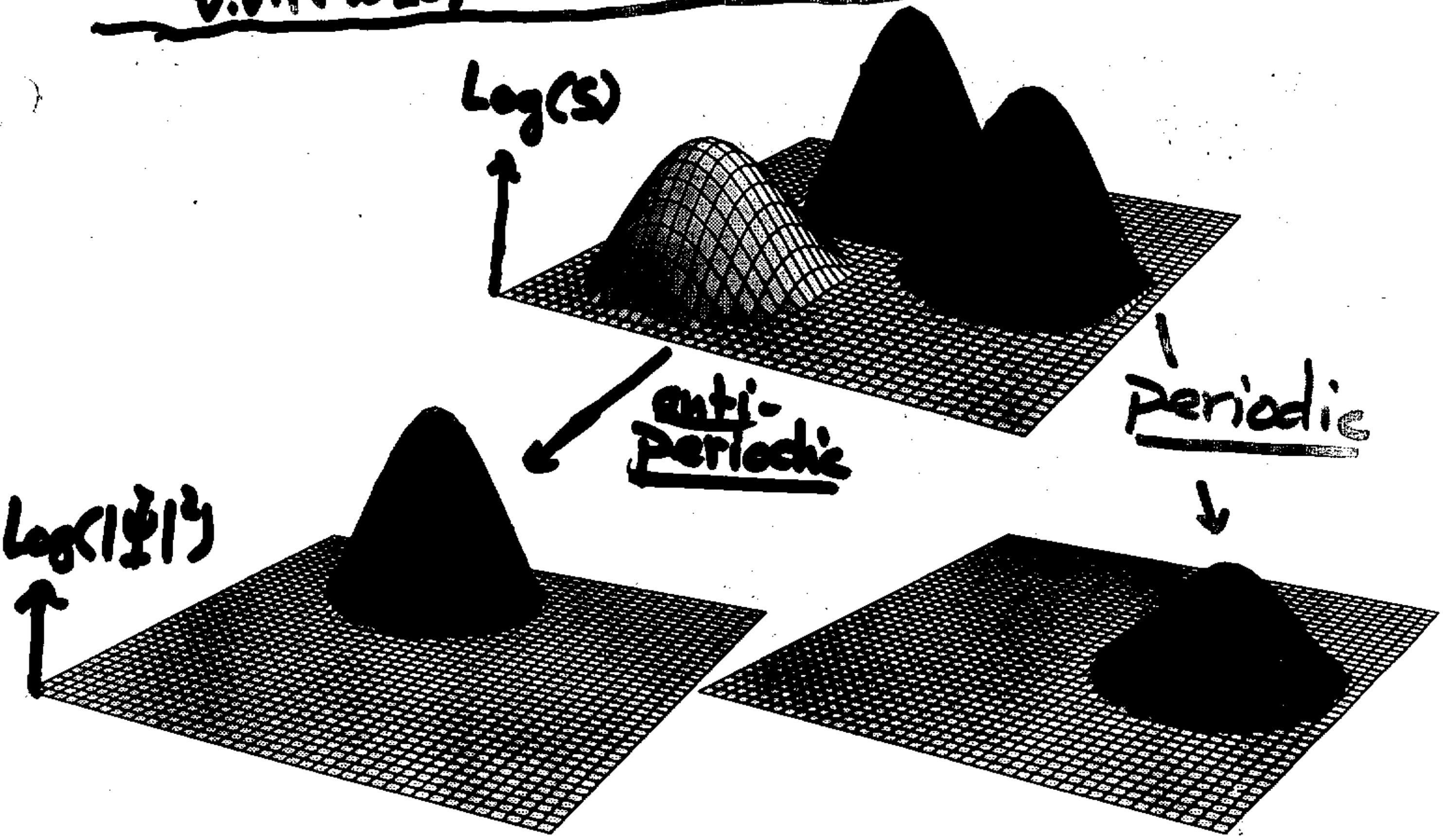


Figure 1: The the action densities (top) for the $SU(3)$ caloron, cut off at $1/(2e)$, on a logarithmic scale, with $(\mu_1, \mu_2, \mu_3) = (-17, -2, 19)/60$ for $t=0$ in the plane defined by $\vec{y}_1 = (-2, -2, 0)$, $\vec{y}_2 = (0, 2, 0)$ and $\vec{y}_3 = (2, -1, 0)$, for $\beta = 1$, with masses $8\pi^2\nu_i$, $(\nu_1, \nu_2, \nu_3) = (0.25, 0.35, 0.4)$. On the bottom-left is shown the zero-mode density for fermions with anti-periodic boundary conditions in time and on the bottom-right for periodic boundary conditions, at equal logarithmic scales, cut off below $1/e^5$.

(with T. Kraan and M. Chernodub)
NPB (Proc. Suppl) 83-84 (2000) 556

Zero-mode periodic up to $\exp(-2\pi i z)$ $\quad \psi_z(\vec{x}, t+\beta) = e^{-2\pi i z}\psi_z(\vec{x}, t)$

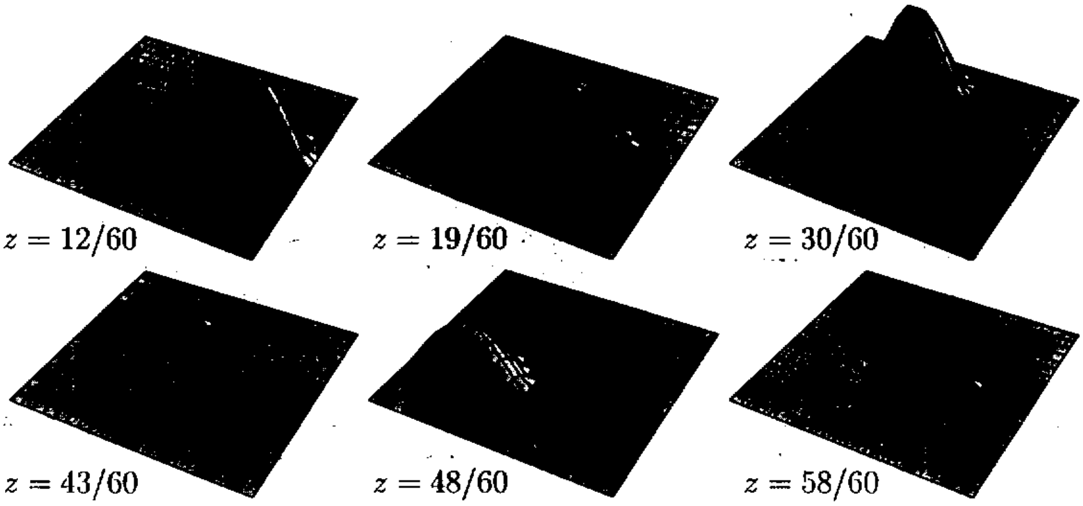


Figure 1: The logarithm of the properly normalized zero-mode density for a typical $SU(3)$ caloron of charge 1, cycling through z. Shown are $z = \mu_j$ (for linear plots see Fig. 2) and three values of z roughly in the middle of each interval $z \in [\mu_j, \mu_{j+1}]$. All plots are on the same scale, cutoff for values of the logarithm below -5. The zero-mode with anti-periodic boundary conditions is found at $z = 30/60$. For the action density of the associated gauge field, see Ref. [13]. See also Ref. [19].

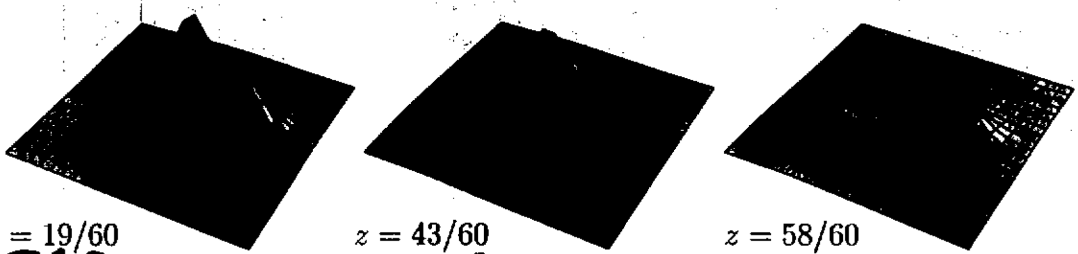


Figure 2: The properly normalized delocalized zero-mode densities for $z = \mu_j$, on the same *linear* scale (cmp. Fig. 1).

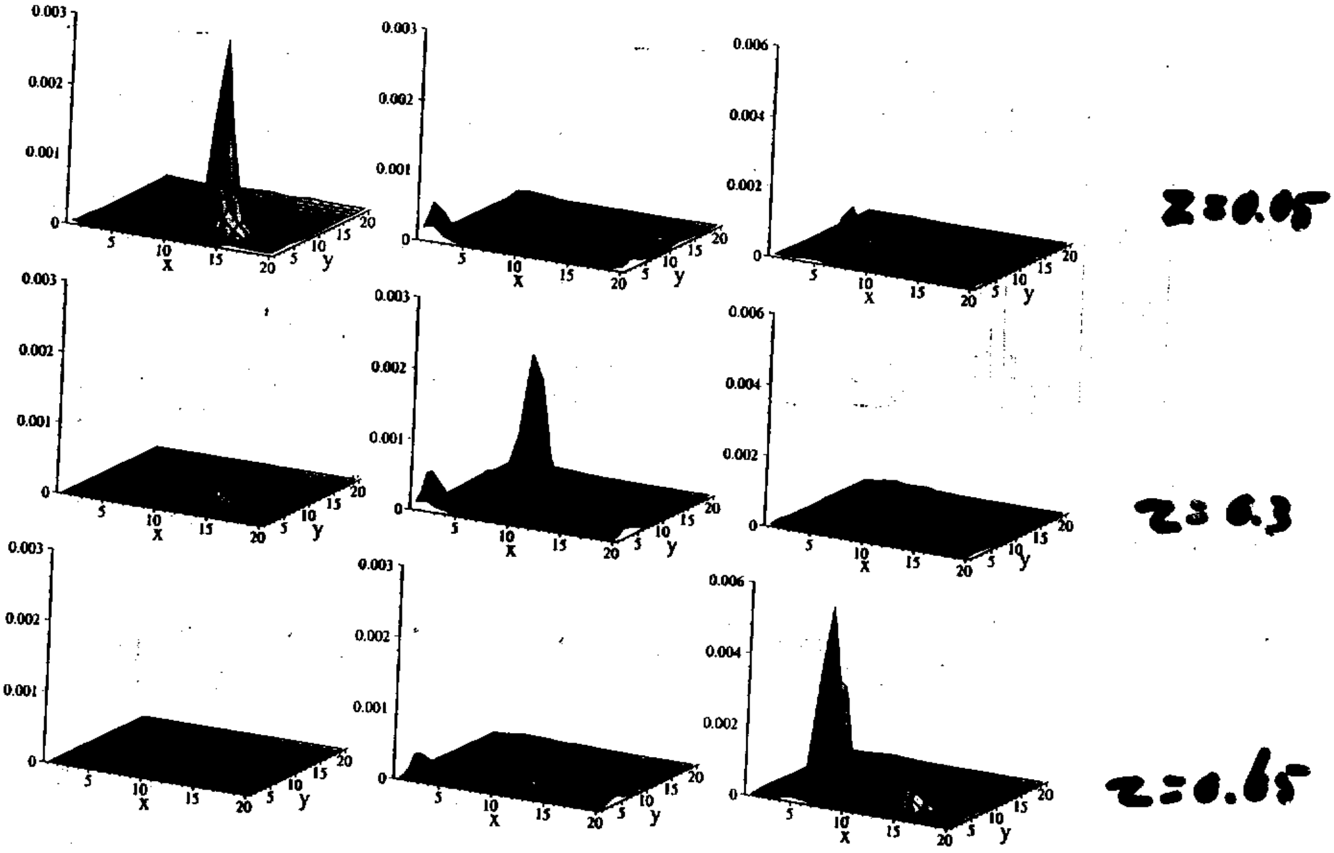


Figure 8: Slices of the scalar density for $6 \times 20^3, \beta = 8.20$, configuration 125. We show x, y-slices at $t = 5$, $z = 9$ (left column), at $t = 2$, $z = 19$ (center column) and $t = 5$, $z = 18$ (right column). The values for ζ are $\zeta = 0.05, 0.3, 0.65$ (from top to bottom).

From: C. Gattringer, S. Schaefer,
Nucl. Phys. B654 (2003) 30, hep-lat/0212029

An SU(2) KvBLL caloron gas model and confinement

P. Gerhold, E.-M. Ilgenfritz and M. Müller-Preussker, NPB 760 (2007) 1

$$A_\mu^{per}(x) = e^{-2\pi i \vec{\omega}\cdot\vec{\tau}} \sum_i A_\mu^{(i),alg}(x)\, e^{2\pi i \vec{\omega}\cdot\vec{\tau}} + 2\pi \vec{\omega}\cdot\vec{\tau}\,\delta_{\mu 4}$$

for a density $n = 1\,\mathrm{fm}^{-4}$ and $\bar\rho = 0.33$ fm good enough (dilute)

$D_1(\rho,T) = A\rho^{b-5}\exp(-c\rho^2)$ ($\bar\rho$ fixed)

$\int D_1(\rho,T)\,d\rho = 1 \quad T \le T_c, \; \omega = \frac{1}{4}\;(\nu = \ldots$

$D_2(\rho,T) = A\rho^{b-5}\exp(-\frac{4}{3}(\pi\rho T)^2)$ ($\bar\rho$ running)

$\int D_2(\rho,T)\,d\rho = 1 \quad T \ge T_c, \; \omega = \alpha(=\nu)$

$\left[b = \frac{11}{3}N_c - \frac{2}{3}N_f = \frac{22}{3} \; (N_c = 2, N_f = 0) \right]$

$\bar\rho(T_c)_{conf} = \bar\rho(T_c)_{deconf} = 0.37$

↳ determines c

$T_c = 178$ MeV , $\sigma(T=0) = 318$ MeV/fm

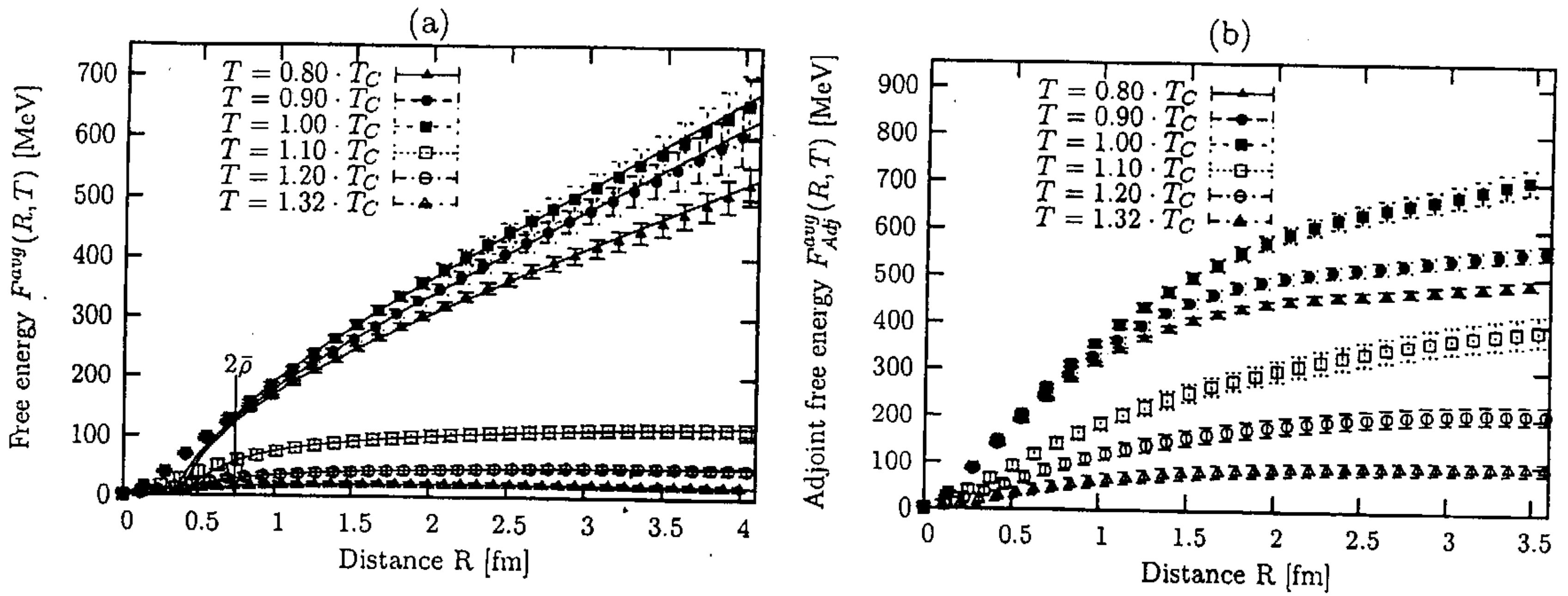


FIG. 11: color averaged free energy versus distance R at different temperatures $T/T_C = 0.8$, 0.9, 1.0 for the confined and at $T/T_C = 1.10$, 1.20, 1.32 for the deconfined phase in the fundamental (a) and in the adjoint (b) representation. The fundamental potentials are fitted to (43) for $R \geq 2\bar{\rho}$. This minimal distance is marked.

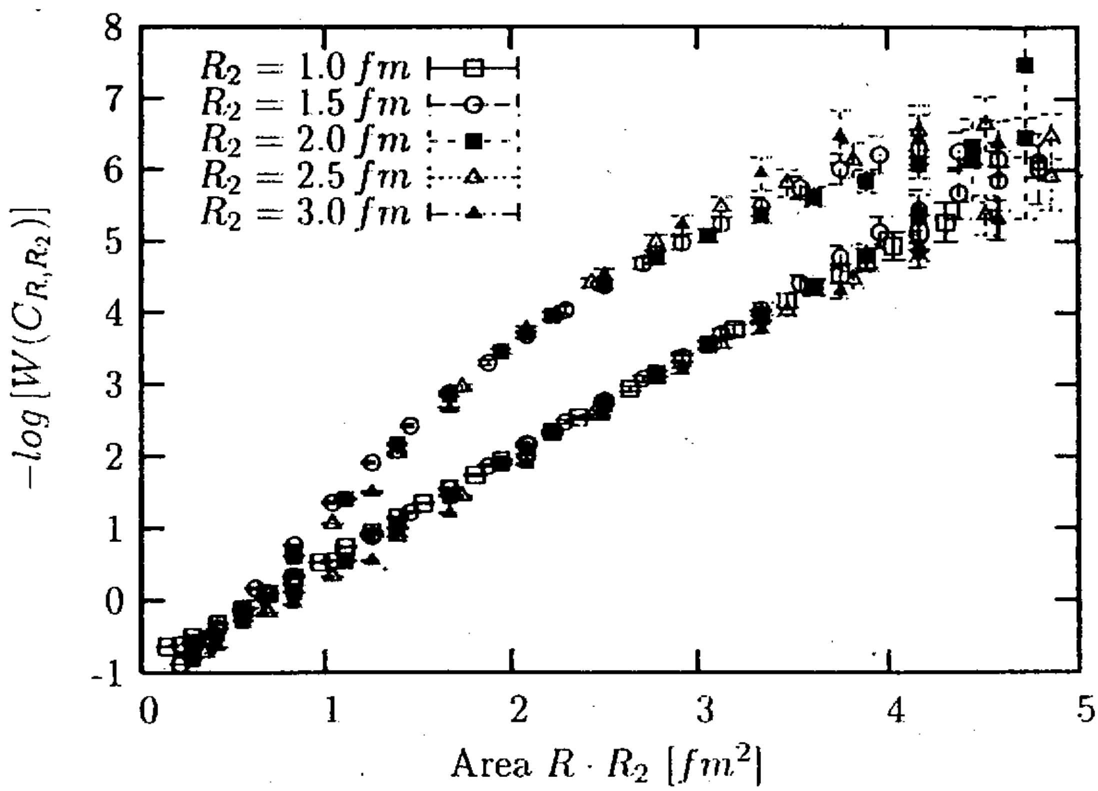
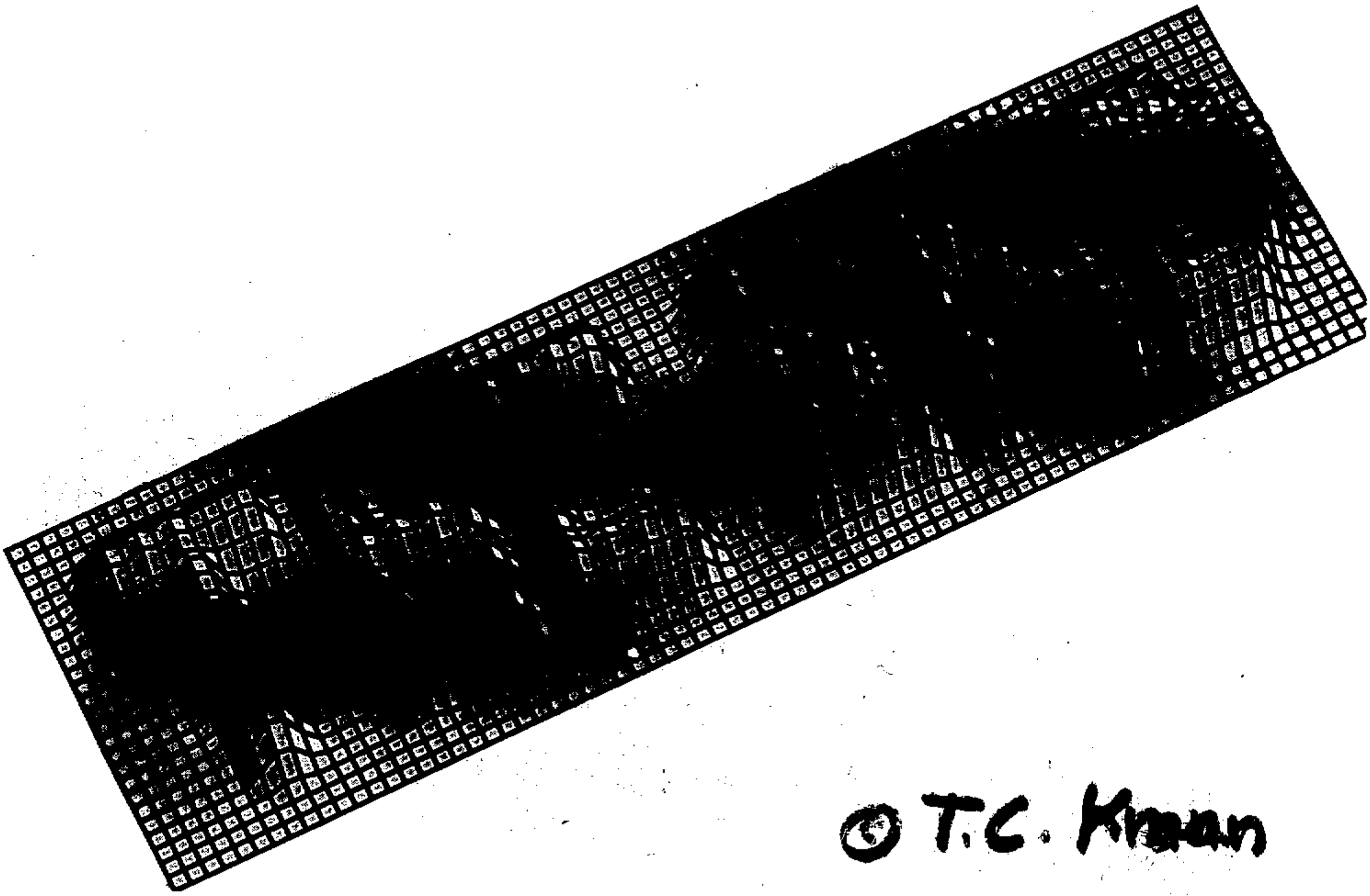


FIG. 8: Negative logarithm of rectangular Wilson loops, $-log(W(C_{R,R_2}))$, with side lengths R, R_2 in fundamental (lower curve) and adjoint representation versus the area $R \cdot R_2$. The different symbols correspond to different side lengths R_2.



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An <u>Atipical</u> $SU(73)$
Calbron <u>solution</u>.