Hamiltonian approach to Yang–Mills theories in Coulomb gauge

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Outline

1. Hamiltonian approach to gauge theories
   - Canonical quantisation
   - Hamilton operator in Coulomb gauge
2. Variational solution of the Yang–Mills Schrödinger equation
   - Vacuum functional
   - Dyson–Schwinger equations
   - Coulomb potential, running coupling and dielectric function
3. The η' mass
   - The $U_A(1)$ puzzle
   - Anomaly contribution to the mass

Starting with the Lagrangian

Electrodynamics

$$\mathcal{L} = \frac{1}{2}(\mathbf{E} \cdot \mathbf{E} - \mathbf{B} \cdot \mathbf{B}),$$

where

$$\mathbf{E} = -\boldsymbol{\nabla} A_0 - \partial_t \mathbf{A}$$

$$\mathbf{B} = \boldsymbol{\nabla} \times \mathbf{A}$$

$\mathcal{L}$ is invariant under the infinitesimal gauge transformation

$$A_\mu \to A_\mu + \frac{1}{g}\, \partial_\mu \phi(x)\, .$$

Starting with the Lagrangian

Yang–Mills theory

$$\mathcal{L} = \frac{1}{2}\left(\mathbf{E}^a \cdot \mathbf{E}^a - \mathbf{B}^a \cdot \mathbf{B}^a\right),$$

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$\mathcal{L}$ is invariant under the infinitesimal gauge transformation

$$A_\mu^a \to A_\mu^a + \frac{1}{g}\left(\delta^{ab}\partial_\mu \qquad\qquad\qquad \right)\phi^b(x)\,,$$

Starting with the Lagrangian

Yang–Mills theory

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where

$$\mathbf{E}^a = -\boldsymbol{\nabla} A_0^a - \partial_t \mathbf{A}^a + g\, f^{abc} \mathbf{A}^b\, A_0^c$$
$$\mathbf{B}^a = \boldsymbol{\nabla} \times \mathbf{A}^a - \frac{g}{2}\, f^{abc} \mathbf{A}^b \times \mathbf{A}^c$$

$\mathcal{L}$ is invariant under the infinitesimal gauge transformation

$$A_\mu^a \to A_\mu^a + \frac{1}{g}\left(\delta^{ab}\partial_\mu + g\, f^{acb}\, A_\mu^c\right)\phi^b(x)\,,$$

Classical Hamiltonian

Take $A_\mu^a = (A_0^a, -\mathbf{A}^a)$ as "coordinates" and derive the momenta

$$\Pi_\mu^a(x) = \frac{\partial \mathcal{S}}{\partial \dot{A}_\mu^a(x)} = \begin{cases} \Pi_0^a = 0 \\ \Pi_i^a = E_i^a \end{cases}$$

After Legendre transformation

$$\mathcal{H} = \int \mathrm{d}^3x \, \frac{1}{2} \left[(E_i^a)^2 + (B_i^a)^2 \right] - \int \mathrm{d}^3x \; A_0^a \left[\hat{D}_i^{ab}[A] E_i^b + g \rho_m^a \right]$$

Covariant derivative: $\hat{D}_\mu^{ab}[A] = \delta^{ab} \partial_\mu + g \hat{A}_\mu^{ab}$

Canonical quantisation

$\Pi_0 = 0 \;\Rightarrow$ difficulty with CCR!

- choose Weyl gauge: $A_0^a = 0$;
- spatial CCRs: $\left[A_i^a(\mathbf{x}), \Pi_j^b(\mathbf{y})\right] = i\delta^{ab}\delta_{ij}\delta(\mathbf{x}-\mathbf{y})$.

Schrödinger equation in Weyl gauge

$$H\Psi[A] = \int \mathrm{d}^3x \, \frac{1}{2}\Big[(\Pi_i^a)^2 + (B_i^a)^2\Big]\Psi[A] = E\Psi[A] \,,$$

where Π_i^a acts as

$$\Pi_i^a(\mathbf{x}) = \frac{1}{i}\,\frac{\delta}{\delta A_i^a(\mathbf{x})} \,.$$

Constrained quantisation

Gauss' law has the form

$$\hat{D}_i^{ab}[A]\,\Pi_i^b + g\rho_m^a = 0$$

- vanishes from the EOM due to the choice of the Weyl gauge
- can not be imposed as operator identity
- has to be imposed as a constraint on the wave functional

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Coulomb gauge: $\partial_i A_i^a(\mathbf{x}) = 0$

The CCRs in Coulomb gauge are

$$\left[A_i^{\perp a}(\mathbf{x}), \Pi_j^{\perp b}(\mathbf{y})\right] = i\delta^{ab} t_{ij}(\mathbf{x})\delta(\mathbf{x}-\mathbf{y}) ,$$

with $t_{ij}(\mathbf{x}) = \delta_{ij} - \partial_i\partial_j/\partial^2$.

$$\int \mathcal{D}A\, \Psi^*[A]\, \Phi[A] \longrightarrow \int \mathcal{D}A^\perp\, \mathcal{J}[A^\perp]\, \Psi^*[A^\perp]\, \Phi[A^\perp]$$

The Faddeev–Popov determinant

$$\mathcal{J}[A^\perp] = \mathrm{Det}(-\partial \cdot \hat{D}[A^\perp])$$

can be interpreted as the Jacobian of the transformation.

Resolution of Gauss' law

$$\left(\hat{D}_i^{ab}[A]\,\Pi_i^b + g\rho_m^a\right)\Psi[A] = 0$$

- split up the momentum operator $\Pi_i = \Pi_i^{\perp} + \Pi_i^{\parallel}$,
- write the longitudinal component as $\Pi_i^{\parallel} = -\partial_i \xi$,
- use $\partial_i \Pi_i^{\perp} = 0$,

$$\Pi_i^{\parallel}\Psi[A^{\perp}] = \partial_i(-\partial \cdot \hat{D}[A^{\perp}])^{-1}\rho_{tot}\Psi[A^{\perp}]\,,$$

where

$$\rho_{tot}^a = \rho_m^a + \hat{A}_j^{\perp ab}\Pi_j^{\perp b}\,.$$

Coulomb gauge Hamiltonian

From electrodynamics. . .

$$H = \int d^3x \frac{1}{2}\left[\Pi_i^{\perp}(\mathbf{x})\,\Pi_i^{\perp}(\mathbf{x}) + B_i(\mathbf{x})B_i(\mathbf{x})\right] +$$

$$+ \frac{g^2}{2} \int d^3x\, d^3y \rho_m(\mathbf{x})\, F(\mathbf{x},\mathbf{y})\, \rho_m(\mathbf{y})$$

$$F(\mathbf{x},\mathbf{y}) = \left[(-\partial^2)^{-1}\right]_{\mathbf{x},\mathbf{y}} = \frac{1}{4\pi|\mathbf{x}-\mathbf{y}|}$$

Coulomb gauge Hamiltonian

. . . to. . .

$$H = \int \mathrm{d}^3x \frac{1}{2}\Big[\Pi_i^{\perp a}(\mathbf{x})\,\Pi_i^{\perp a}(\mathbf{x}) + B_i^a(\mathbf{x}) B_i^a(\mathbf{x})\Big] +$$

$$+\frac{g^2}{2}\int \mathrm{d}^3x\,\mathrm{d}^3y\,\rho_m(\mathbf{x})\,F(\mathbf{x},\mathbf{y})\,\rho_m(\mathbf{y})$$

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Coulomb gauge Hamiltonian

. . . to. . .

$$H = \int d^3x \frac{1}{2}\left[\mathcal{J}^{-1}[A^\perp]\,\Pi_i^{\perp a}(\mathbf{x})\,\mathcal{J}[A^\perp]\,\Pi_i^{\perp a}(\mathbf{x}) + B_i^a(\mathbf{x})B_i^a(\mathbf{x})\right] +$$

$$+\frac{g^2}{2}\int d^3x\, d^3y\, \rho_m(\mathbf{x})\, F(\mathbf{x},\mathbf{y})\, \rho_m(\mathbf{y})$$

$$F(\mathbf{x},\mathbf{y}) = \left[(-\partial^2)^{-1}\right]_{\mathbf{x},\mathbf{y}} = \frac{1}{4\pi|\mathbf{x}-\mathbf{y}|}$$

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$$+\frac{g^2}{2}\int \mathrm{d}^3x\,\mathrm{d}^3y\,\rho^a_{tot}(\mathbf{x})F(\mathbf{x},\mathbf{y})\rho^b_{tot}(\mathbf{y})$$

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Coulomb gauge Hamiltonian

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$$+\frac{g^2}{2}\int d^3x\, d^3y\, \mathcal{J}^{-1}[A^\perp]\rho_{tot}^a(\mathbf{x})\mathcal{J}[A^\perp]F(\mathbf{x},\mathbf{y})\rho_{tot}^b(\mathbf{y})$$

$$F(\mathbf{x},\mathbf{y}) = \Big[(-\partial^2)^{-1}\Big]_{\mathbf{x},\mathbf{y}} = \frac{1}{4\pi|\mathbf{x}-\mathbf{y}|}$$

Coulomb gauge Hamiltonian

. . . Yang–Mills theory

$$H = \int d^3x \frac{1}{2}\left[\mathcal{J}^{-1}[A^\perp]\,\Pi_i^{\perp a}(\mathbf{x})\,\mathcal{J}[A^\perp]\,\Pi_i^{\perp a}(\mathbf{x}) + B_i^a(\mathbf{x})B_i^a(\mathbf{x})\right] +$$

$$+\frac{g^2}{2}\int d^3x\, d^3y\, \mathcal{J}^{-1}[A^\perp]\rho^a_{tot}(\mathbf{x})\mathcal{J}[A^\perp]F^{ab}[A^\perp](\mathbf{x},\mathbf{y})\rho^b_{tot}(\mathbf{y})$$

$$F^{ab}[A^\perp](\mathbf{x},\mathbf{y}) = \left[\left(-\partial\cdot\hat{D}[A^\perp]\right)^{-1}\left(-\partial^2\right)\left(-\partial\cdot\hat{D}[A^\perp]\right)^{-1}\right]^{ab}_{\mathbf{x},\mathbf{y}}$$

N. H. Christ and T. D. Lee, PRD**22**, 939 (1980)

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Variational approach

For a given functional Ψ_ω depending on ω, minimise

$$E[\omega] = \frac{\langle \Psi_\omega | \, H \, | \Psi_\omega \rangle}{\langle \Psi_\omega | \Psi_\omega \rangle} \longrightarrow \text{min}$$

Gaussian functional

$$\Psi_\omega[A] = \mathcal{N} \, \mathcal{J}^{-\frac{1}{2}} \, \exp\left\{ -\frac{1}{2} \int \mathrm{d}^3x \, \mathrm{d}^3y \, A_i^a(\mathbf{x}) \, \omega_{ij}(\mathbf{x},\mathbf{y}) \, A_j^a(\mathbf{y}) \right\}$$

C. Feuchter and H. Reinhardt, PRD**70**, 105021 (2004)

Basic propagators

Gluon propagator

$$F.T.\langle\Psi_\omega|\, A_i^a(\mathbf{x}) A_j^b(\mathbf{y})\, |\Psi_\omega\rangle = \frac{\delta^{ab}\, t_{ij}(\mathbf{p})}{2\,\omega(\mathbf{p})}$$

Ghost propagator

$$F.T.\langle\Psi_\omega|\, \left[(-\partial\cdot\hat{D})^{-1}\right]^{ab}_{\mathbf{x},\mathbf{y}}\, |\Psi_\omega\rangle =: \delta^{ab}\,\frac{d(\mathbf{p})}{g\,\mathbf{p}^2}$$

The minimization of the vacuum energy density yields a system of coupled integral equations for $\omega(\mathbf{p})$, $d(\mathbf{p})$.

Dyson–Schwinger equations: IR expansion

Assumption: power laws

$$\omega(\mathbf{p} \to 0) \sim \frac{A}{(\mathbf{p}^2)^\alpha}\,, \quad d(\mathbf{p} \to 0) \sim \frac{B}{(\mathbf{p}^2)^\kappa}$$

IR expansion of the integrals yields

$$2\kappa - \alpha = \frac{1}{2}\,, \qquad \frac{A}{B^2} = \frac{1}{6\pi^2}\,\frac{\kappa+1}{\kappa(2\kappa+1)}$$

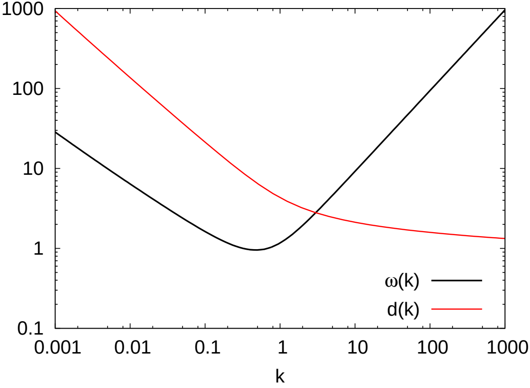
and an equation for κ which has two solutions

$$\kappa = 0.398\,,\ \alpha = 0.296 \qquad \text{and} \qquad \kappa = \frac{1}{2}\,,\ \alpha = \frac{1}{2}$$

W. Schleifenbaum, M. Leder and H. Reinhardt, PRD**73**, 125019 (2006)

Dyson–Schwinger equations: full numerical solution

Solution with $\kappa = 0.398$

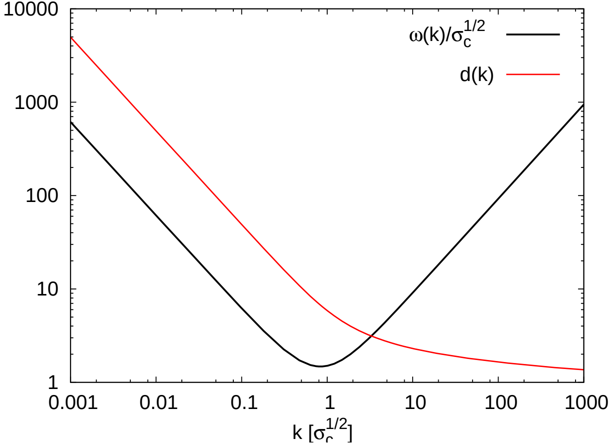


C. Feuchter and H. Reinhardt, PRD**70**, 105021 (2004)

Same infrared exponents obtained also from the ERGE, see talk by Markus Leder.

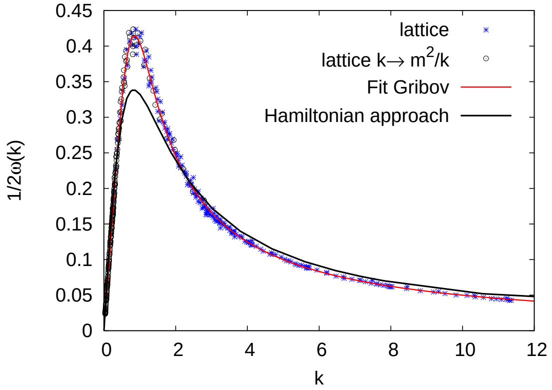
Coupled integral equations: full numerical solution

Solution with $\kappa = 1/2$



D. Epple, H. Reinhardt and W. Schleifenbaum, PRD**75**, 045011 (2007)

Comparison with lattice data



Gribov's formula

$$\omega(\mathbf{k}) = \sqrt{\mathbf{k}^2 + \frac{m^4}{\mathbf{k}^2}}$$

$$m = 0.88 \text{ GeV}$$

G. Burgio, M. Quandt and H. Reinhardt, arXiv:0807.3291 [hep-lat] (2008)

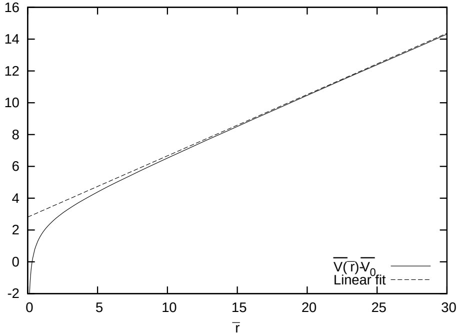
Coulomb potential

Coulomb potential in momentum space

$$V(\mathbf{k}) \sim \frac{d^2(\mathbf{k})}{\mathbf{k}^2}$$

$\kappa = 1/2 \Rightarrow$ linearly rising potential

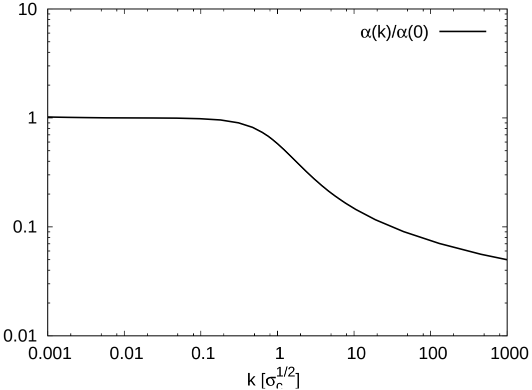
$$V(r \to \infty) \sim \sigma_c \, r$$



D. Epple, H. Reinhardt and W. Schleifenbaum, PRD**75**, 045011 (2007)

Running coupling

From the nonrenormalisation of the ghost-gluon vertex it is possible to define a running coupling



D. Epple, H. Reinhardt and W. Schleifenbaum, PRD**75**, 045011 (2007)

Dielectric function

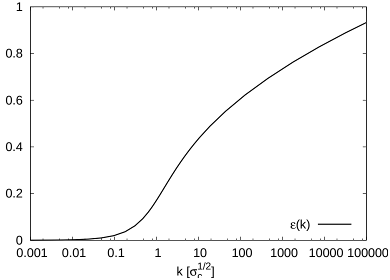
Electrodynamics

$$E_i^{\|} = \partial_i (-\partial^2)^{-1} \rho_m \, \varepsilon^{-1}$$

Yang–Mills theory

$$E_i^{\|a} = \langle \Pi_i^{\|a} \rangle = \partial_i \, d \, (-\partial^2)^{-1} \rho_m^a$$

$$\varepsilon(\mathbf{k}) = d^{-1}(\mathbf{k})$$



H. Reinhardt, PRL**101**, 061602 (2008)

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Axial anomaly and $U_A(1)$ problem

How is the $U_A(1)$ symmetry realised in nature?

Wigner-Weyl $\longrightarrow$ degenerate parity doublets

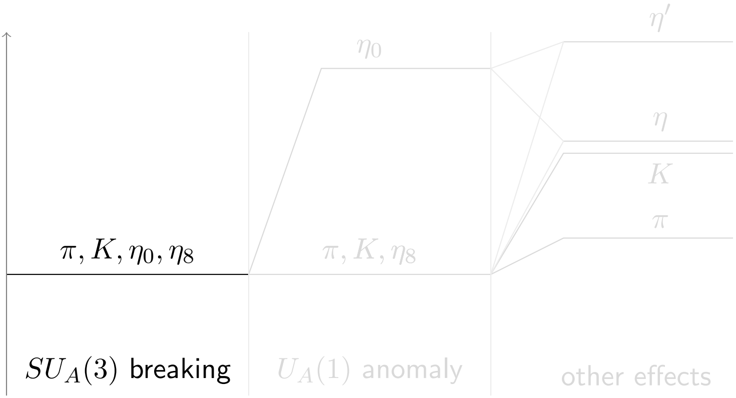
Nambu-Goldstone $\longrightarrow$ light particle with $m \leqslant \sqrt{3}\, m_\pi$

Solution of the problem

- the axial symmetry is anomalously broken
- the anomaly term is a total derivative
- despite this, it has non-perturbative effects

$$\mathcal{L}_{YM} = \frac{1}{2}\left(\mathbf{E}^a \cdot \mathbf{E}^a - \mathbf{B}^a \cdot \mathbf{B}^a\right) + \theta\, \frac{g^2}{8\pi^2}\, \mathbf{E}^a \cdot \mathbf{B}^a$$

Chiral symmetry breaking pattern



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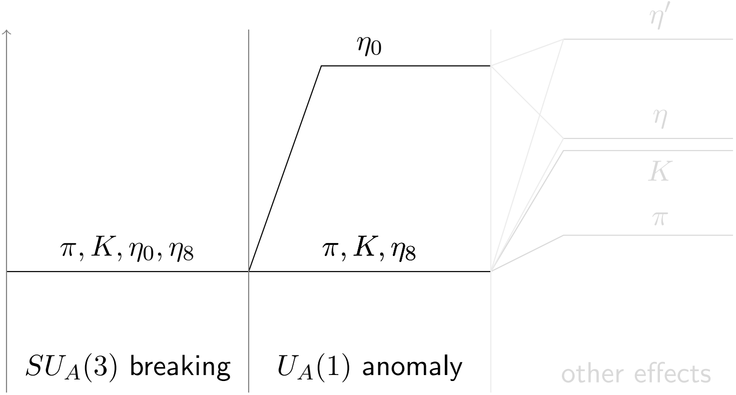
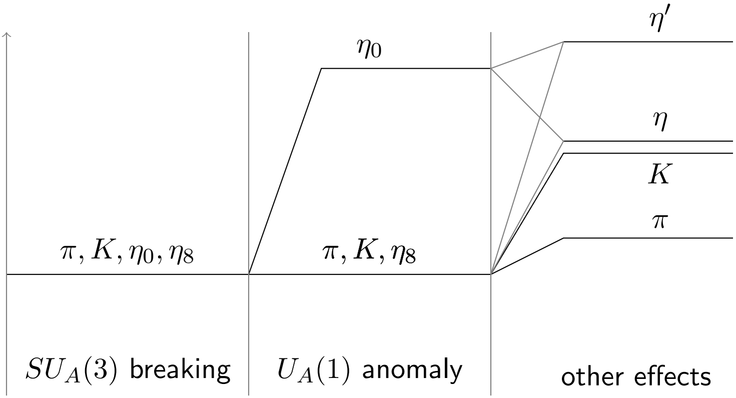


Chiral symmetry breaking pattern



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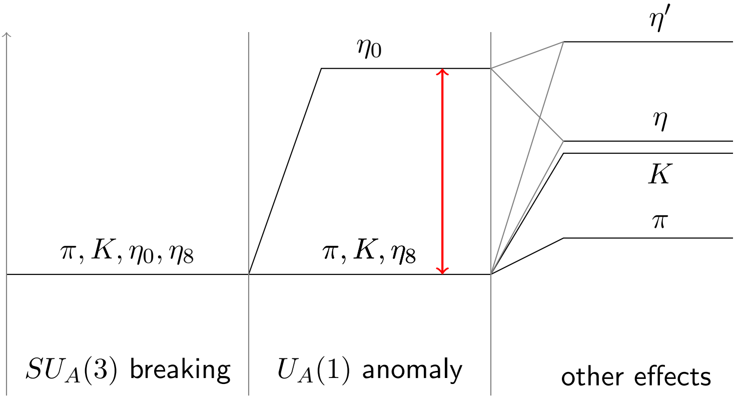

The Witten-Veneziano formula

$$m_{\eta_0}^2 = m_{\eta'}^2 + m_\eta^2 - 2m_K^2 = \frac{2N_f}{F_\pi^2}\,\chi$$

E. Witten, NP**B156**, 269 (1979)

G. Veneziano, NP**B159**, 213 (1979)

Relevance for our model

It can be shown that χ vanishes identically in any order perturbation theory. It is then a crucial quantity to test a non-perturbative approach to Yang–Mills theory.

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Topological susceptibility in Hamiltonian formalism

The Hamilton operator with θ-terms takes the following form

$$H_\theta = H_0 + \theta \, \frac{g^2}{8\pi^2} \, H_1 + \left(\theta \, \frac{g^2}{8\pi^2} \right)^2 H_2$$

with

$$H_1 \sim B_i^a \, \Pi_i^a \; , \qquad H_2 \sim B_i^a \; B_i^a \; .$$

$$\chi = \frac{1}{V} \; \frac{\mathrm{d}^2 \left\langle H_\theta \right\rangle}{\mathrm{d}\theta^2} \bigg|_{\theta=0}$$

Hamiltonian with θ-terms

Since $|\theta| < 10^{-10}$ we use perturbation theory. As we need the second derivative w.r.t. θ we go up to

- first order for H_2 (quadratic in θ), and
- second order for H_1 (linear in θ).

$$\chi = \frac{2}{V} \left(\frac{g^2}{8\pi^2} \right)^2 \left[\langle 0| H_2 |0\rangle - \sum_{n\neq 0} \frac{|\langle 0| H_1 |n\rangle|^2}{E_n} \right]$$

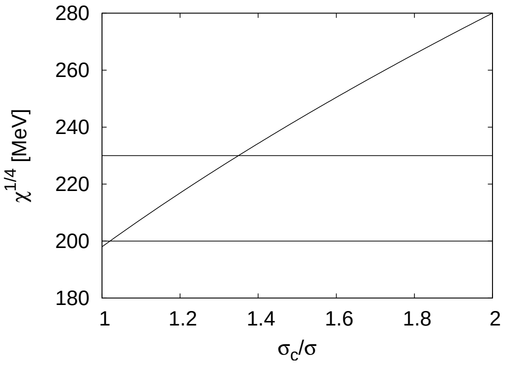
Topological susceptibility in $SU(2)$

Our result

$$\chi = \left[\frac{\sigma_c}{\sigma}\right]^2 (198\ \text{MeV})^4$$

Lattice

$$\chi = (200\text{–}230\ \text{MeV})^4$$



D. R. Campagnari and H. Reinhardt, PRD, in press.

Conclusions

Summary

- Hamiltonian approach yields picture of confinement
- agreement with lattice data
- topological susceptibility has been calculated

Outlook

- insertion of quarks
- improvement of wave functional
- finite temperatures