Hamiltonian Renormalisation Flows of Yang-Mills-Theory in Coulomb Gauge

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Outline

Introduction

- Development of QED: Perturbative renormalisation
- Kadanoff: Block spin transformations in statistical physics
- Wilson: renormalisation group in statistical physics
- Polchinski: Proof of perturbative renormalisability using Wilson's RG
- Wetterich: Introduction of the average effective action: non-perturbative renormalisation: ERGE

  C. Wetterich, Phys.Lett.B301:90-94,1993

  Application: e.g. possible non-perturbative renormalisability of GRT: Quantum Einstein Gravity (Reuter)

The Generating Functional Z_k

$$Z[j] = \int \mathcal{D}\chi \, e^{-S[\chi]+j\cdot\chi}$$

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$$Z_k[j] = \int \mathcal{D}_\Lambda \chi \, e^{-S[\chi] - \Delta S_k[\chi] + j \cdot \chi}$$

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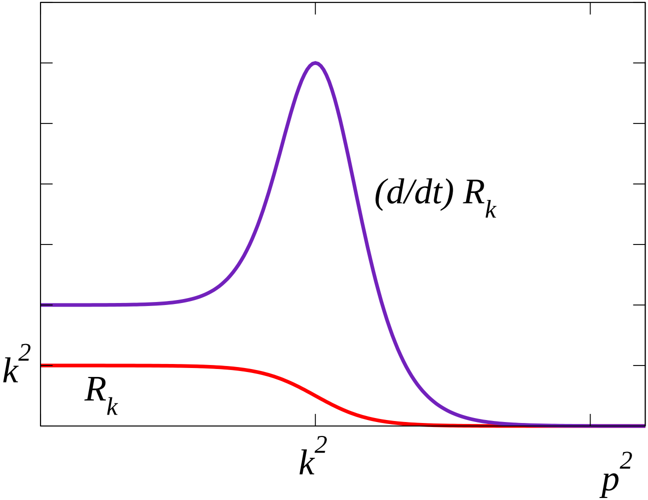
$$\Delta S_k[\chi] = \frac{1}{2} \chi \cdot R_k \cdot \chi$$

IR-regularisation (mass term): $\lim_{p^2/k^2 \to 0} R_k(p) > 0$

recovering full theory: $\lim_{k^2/p^2 \to 0} R_k(p) = 0$

input: bare action $(\Gamma_{\Lambda\to\infty} \to S)$ $\lim_{k^2 \to \Lambda \to \infty} R_k(p) \to \infty$

The Regulator $R_k(p)$



$Z_k \longrightarrow W_k \longrightarrow \Gamma_k$: The Average Effective Action Γ_k

$$Z[j] = e^{W[j]}$$

$$\Gamma[\phi] = -W[j] + j \cdot \phi$$

with j such that $\phi = \frac{\delta W[j]}{\delta j}$

$Z_k \longrightarrow W_k \longrightarrow \Gamma_k$: The Average Effective Action Γ_k

$$Z_k[j] = e^{W_k[j]}$$

$$\Gamma_k[\phi] = -W_k[j_k] + j_k \cdot \phi - \Delta S_k[\phi]$$

with j_k such that $\phi = \frac{\delta W_k[j_k]}{\delta j}$

The ERGE for Γ_k

The Exact Renormalisation Group Equation

$$\partial_t \Gamma_k[\phi] = \frac{1}{2} Tr \left[\partial_t R_k \left(\Gamma_k^{(2)}[\phi] + R_k \right)^{-1} \right]$$

where $\Gamma_k^{(2)}[\phi] = \dfrac{\delta^2 \Gamma_k[\phi]}{\delta\phi\,\delta\phi}$ and $k\dfrac{d}{dk} =: \partial_t$

- Γ_k interpolates between $\Gamma_{k=\Lambda} = S_{bare}$ and $\Gamma_{k=0} = \Gamma$.
- Physics for $k > \Lambda$ is regarded as included already in Γ_Λ.
- Therefore, cutoff Λ is NOT taken to ∞, Λ stays large but finite.

Approximation schemes

Truncations should be systematic and consistent.

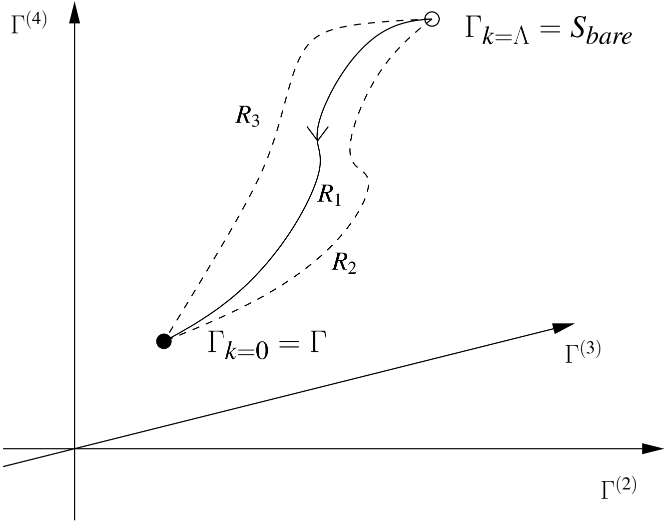
- Derivative expansion

$$\Gamma_k[\phi] = \int d^d x \left\{ U_k(\phi) + \frac{1}{2} Z_k(\phi)(\partial_\mu \phi)^2 + \mathcal{O}(\partial^4) \right\}$$

- Vertex expansion

$$\Gamma_k[\phi] = \sum_{n=0}^{\infty} \frac{1}{n!} \int d^d x_1 \dots d^d x_n \Gamma_k^{(n)}(x_1, \dots, x_n) \phi(x_1) \dots \phi(x_n)$$

Theory Space



Example: Propagator flow

$$\left.\frac{\delta^2}{\delta\phi\,\delta\phi}\right|_{\phi=0}\partial_t\Gamma_k[\phi] = \left.\frac{\delta^2}{\delta\phi\,\delta\phi}\right|_{\phi=0}\frac{1}{2}Tr\left[(\partial_t R_k)\left(\Gamma_k^{(2)}[\phi]+R_k\right)^{-1}\right]$$

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$$\Rightarrow \partial_t \Gamma_k^{(2)} = Tr\left\{(\partial_t R_k)[\Gamma_k^{(2)} + R_k]^{-1}\Gamma_k^{(3)}[\Gamma_k^{(2)} + R_k]^{-1}\Gamma_k^{(3)}[\Gamma_k^{(2)} + R_k]^{-1}\right\}$$
$$- \frac{1}{2} Tr\left\{(\partial_t R_k)[\Gamma_k^{(2)} + R_k]^{-1}\Gamma_k^{(4)}[\Gamma_k^{(2)} + R_k]^{-1}\right\}$$

1. Introduction
2. The Exact Renormalisation Group Equation (ERGE)
3. Hamiltonian RG-flow for YMT in Coulomb Gauge
4. Summary

Definition of Z_k

- Define the theory in an already gauge fixed way: Weyl gauge ($A_0^a = 0$) and Coulomb gauge ($\partial_i A_i^a = 0$).
- S is no action but part of the vacuum wave functional

$$Z_k[J,\sigma,\bar{\sigma}] = \int \mathcal{D}[A,c,\bar{c}]\exp[-S - \Delta S_k + J\cdot A + \bar{\sigma}\cdot c + \bar{c}\cdot\sigma]$$
$$\Delta S_k = \tfrac{1}{2} A\cdot R_k \cdot A + \bar{c}\cdot \tilde{R}_k \cdot c$$

- ΔS_k breaks gauge invariance ⇒ counterterms needed!
- VEV is taken wrt. the full ground state.

$$Z_k[J,\sigma,\bar{\sigma}] = \langle\psi|\exp[-\Delta S_k + J\cdot A + \bar{\sigma}\cdot c + \bar{c}\cdot\sigma]|\psi\rangle$$

- No ansatz for the vacuum wave functional is used.
- No Hamiltonian is used.

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The flow equation for Γ_k

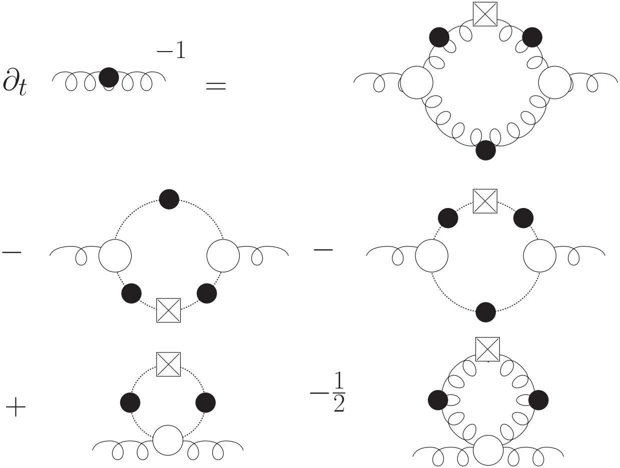
$$Z_k = e^{W_k} \; ; \; \Gamma_k = -W_k + J_k \cdot A + \bar{\sigma}_k \cdot c + \bar{c} \cdot \sigma_k - \frac{1}{2} A \cdot R_k \cdot A - \bar{c} \cdot \tilde{R}_k \cdot c$$

$$A = \frac{\delta W_k}{\delta j} \qquad c = \frac{\delta W_k}{\delta \bar{\sigma}} \qquad \bar{c} = -\frac{\delta W_k}{\delta \sigma}$$

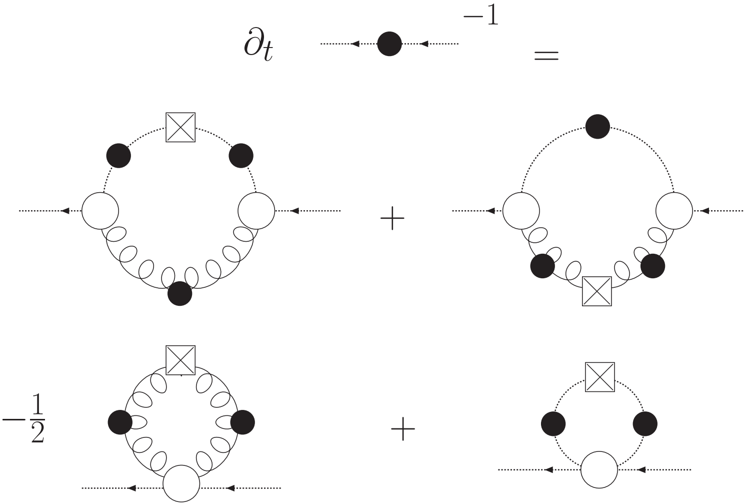
The Exact Renormalisation Group Equation

$$\partial_t \Gamma_k = \frac{1}{2} Tr \left[(\partial_t R_k) \left(\frac{\delta^2 \Gamma_k}{\delta A \delta A} + R_k \right)^{-1} \right] - Tr \left[(\partial_t \tilde{R}_k) \left(\frac{\delta^2 \Gamma_k}{\delta c \delta \bar{c}} + \tilde{R}_k \right)^{-1} \right]$$

The Flow of the Gluon Propagator



The Flow of the Ghost Propagator



Truncation of the Theory Space

Restriction of theory space to

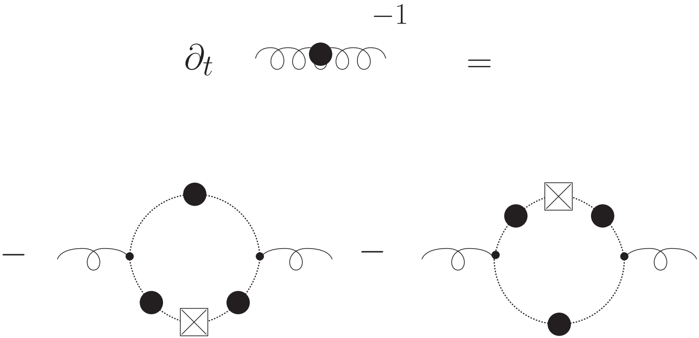
- gluon propagator
- ghost propagator

This would result in trivial flow equations.
$\Rightarrow$ Inclusion of one further operator:

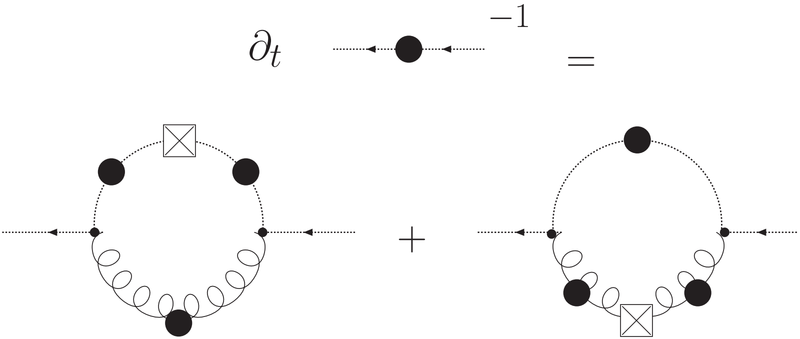
- BARE ghost-gluon vertex

$\Rightarrow$ flow equations decoupled from the rest of theory space

Truncated Gluon Flow



Truncated Ghost Flow



Decomposition

Weyl gauge ⇒ only spatial fields $A^a_{x,y,z}$, $A^a_0 = 0$
Coulomb gauge ⇒ transversality $t_{ij}(\mathbf{p})$ of gluon propagator

$$\frac{\delta^2\Gamma_k[0]}{\delta A^a_i(\mathbf{p})\ \delta A^b_j(\mathbf{q})} = \delta^{ab} t_{ij}(\mathbf{p}) 2\omega_k(p)(2\pi)^d\delta^d(\mathbf{p}+\mathbf{q})$$

$$\frac{\delta^2\Gamma_k[0]}{\delta c^a(\mathbf{p})\ \delta\bar{c}^b(\mathbf{q})} = \delta^{ab}\frac{p^2}{d_k(p)}(2\pi)^d\delta^d(\mathbf{p}+\mathbf{q})$$

$$\frac{\delta^3\Gamma_\Lambda}{\delta A^a_i(\mathbf{p})\ \delta c^b(\mathbf{q})\ \delta\bar{c}^c(\mathbf{k})} = f^{cab}(iq_j)t_{ij}(\mathbf{p})(2\pi)^d\delta^d(\mathbf{p}+\mathbf{q}+\mathbf{k})$$

Flow Equation for ω_k

$$\partial_t \omega_k(p) = -\frac{N_c}{d-1}\int \frac{d^d r}{(2\pi)^d}\partial_t \tilde{R}_k(r)\left(\frac{r^2}{d_k(r)} + \tilde{R}_k(r)\right)^{-2} \cdot \cdot \left(\frac{|\mathbf{r}+\mathbf{p}|^2}{d_k(|\mathbf{r}+\mathbf{p}|)} + \tilde{R}_k(|\mathbf{r}+\mathbf{p}|)\right)^{-1}\left(r^2 - (\hat{\mathbf{p}}\cdot\mathbf{r})^2\right)$$

Flow Equation for d_k

$$\partial_t d_k^{-1}(p) = -N_c \int \frac{d^d r}{(2\pi)^d} \cdot$$

$$\cdot \left[\partial_t R_k(r) [2\omega_k(r) + R_k(r)]^{-2} \left(\frac{|\mathbf{r}+\mathbf{p}|^2}{d_k(|\mathbf{r}+\mathbf{p}|)} + \tilde{R}_k(|\mathbf{r}+\mathbf{p}|) \right)^{-1} \cdot \ldots \right.$$

$$\left. + \partial_t \tilde{R}_k(r) \left(\frac{r^2}{d_k(r)} + \tilde{R}_k(r) \right)^{-2} [2\omega_k(|\mathbf{r}+\mathbf{p}|) + R_k(|\mathbf{r}+\mathbf{p}|)]^{-1} \cdot \ldots \right]$$

Solving Differential Equations ?

- RG-equations are two coupled 1^{st} order ODE in k.
- Possible solution: Integrating numerically from initial conditions $[\omega_\Lambda(p), d_\Lambda(p)]$ down to $[\omega_0(p), d_0(p)]$ (e.g. using Runge-Kutta).
- Initial conditions: $[\omega_\Lambda(p) = p, d_\Lambda(p) = 1]$

Problems with the Differential Equations

How to implement renormalisation conditions?

- Horizon condition: $d_0^{-1}(0) = 0$
- Asymptotic freedom: $\omega_0(p) \sim p$ for $p \sim \Lambda$

$\Rightarrow$ Counterterms for ω_Λ and d_Λ^{-1} are needed:

- $\omega_\Lambda(p) = -\alpha - \beta p + p$
- $d_\Lambda^{-1}(p) = const.$

$\Rightarrow$ How to choose the c.t. to satisfy the renormalisation conditions?

Solving Integral Equations !

Integrate the differential flow equations:

$$\omega_q(p) - \omega_\Lambda(p) = \int_\Lambda^q dk \,\text{loop}(k,p)[d_k]$$

$$d_q^{-1}(p) - d_\Lambda^{-1}(p) = \int_\Lambda^q dk \,\text{loop}(k,p)[\omega_k, d_k]$$

$\Rightarrow$ Iteration of $\omega_q(p)$ and $d_q(p)$.

Fine Tuning

- Gluon equation

$$p \sim \Lambda : \; \omega_0(p) - \omega_\Lambda(p) = \underbrace{\int_\Lambda^0 dk \, \text{loop}(k,p)[d_k]}_{\text{fit } \alpha+\beta p \text{ for } p\sim\Lambda}$$

With $\omega_\Lambda(p) = -\alpha + (1-\beta)p \Rightarrow \omega_0(p) = p$ for $p \sim \Lambda$

- Ghost equation

$$\underbrace{d_0^{-1}(0)}_{\overset{!}{=}0} - d_\Lambda^{-1}(0) \; = \int_\Lambda^0 dk \, \text{loop}(k,p=0)[\omega_k, d_k]$$

$$\Rightarrow \text{ Choose } d_\Lambda^{-1}(p) = -\int_\Lambda^0 dk \, \text{loop}(k,0)[\omega_k, d_k]$$

Approximation $\omega_k \longrightarrow \omega_0$, $d_k \longrightarrow d_0$ in the loop

$\Rightarrow$ Analytical integration of flow integrals feasible

- Gluon equation

$$\omega_0(p) - \omega_\Lambda(p) = \frac{N_c}{2(d-1)} \left[\int \frac{d^d r}{(2\pi)^d} \left(\frac{r^2}{d_0(r)} + \tilde{R}_k(r) \right)^{-1} \cdot \right.$$

$$\left. \cdot \left(\frac{|\mathbf{r}+\mathbf{p}|^2}{d_0(|\mathbf{r}+\mathbf{p}|)} + \tilde{R}_k(|\mathbf{r}+\mathbf{p}|) \right)^{-1} \cdot \ldots \right]_{k=\Lambda}^{k=0}$$

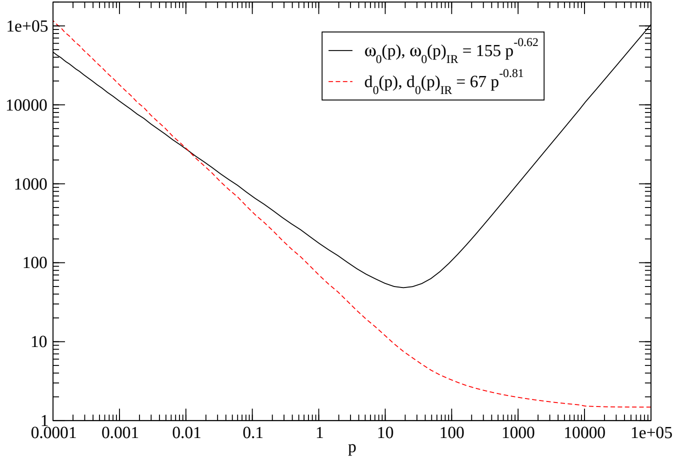
Approximation $\omega_k \longrightarrow \omega_0 \ , \ d_k \longrightarrow d_0$ in the loop

- Ghost equation

$$\begin{aligned} d_0^{-1}(p) - d_\Lambda^{-1}(p) = & N_c \left[\int \frac{d^d r}{(2\pi)^d} (2\omega_0(r) + R_k(r))^{-1} \cdot \right. \\ & \left. \cdot \left(\frac{|\mathbf{r}+\mathbf{p}|^2}{d_0(|\mathbf{r}+\mathbf{p}|)} + \tilde{R}_k(|\mathbf{r}+\mathbf{p}|) \right)^{-1} \cdot \ldots \right]_{k=\Lambda}^{k=0} \end{aligned}$$

⇒ Integrated RG equations correspond to DSE.

Results for Approximate Flow Equation



1000 iterations, start values $\omega_0(p) = d_0(p) = \text{const.}$

Λ=1e4, relax=0.5, cheb-nodes = 100, gau-leg-nodes=100

Results for Approximate Flow Equation

$$\omega_0(p)_{IR} \sim p^{-0.62} \quad d_0(p)_{IR} \sim p^{-0.81}$$

This result is close to one of two possible solutions found by IR-analysis of DSE and a numerical calculation resp. in

W. Schleifenbaum, M. Leder, H. Reinhardt, PRD **73** :125019,2006

C. Feuchter, H. Reinhardt, PRD **70**:105021,2004

Another possible solution

$$\omega_0(p)_{IR} \sim p^{-1} \quad d_0(p)_{IR} \sim p^{-1}$$

found by IR-analysis of DSE and a numerical calculation resp. in

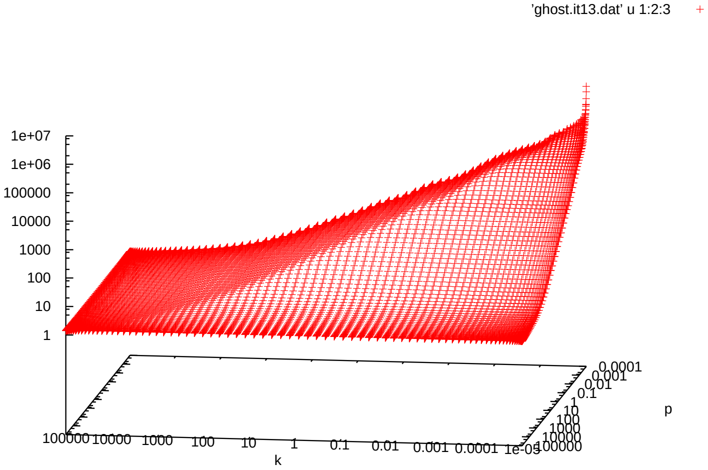
W. Schleifenbaum, M. Leder, H. Reinhardt, PRD **73** :125019,2006

D. Epple, H. Reinhardt, W. Schleifenbaum, PRD **75**:045011,2007

is not seen here.

Flow with Full k-Dependence (PRELIMINARY)

Flow of the ghost form factor $d_k(p)$



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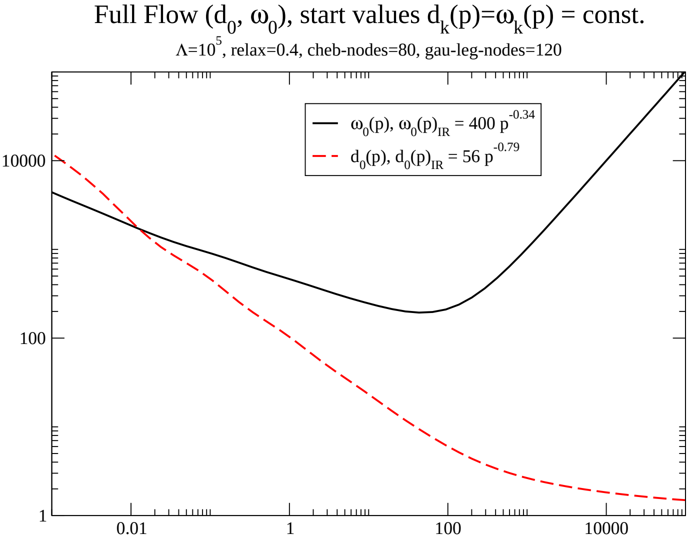

Results for Full Flow (PRELIMINARY)

- Ghost

$$d_0(p)_{IR} = 56\, p^{-0.79}$$

Previously found IR-behaviour is reproduced.

- Gluon

$$\omega_0(p)_{IR} = 400\, p^{-0.34}$$

Weaker IR-divergence than previously found.

$\Rightarrow$ Improval of the numerics necessary
$\Rightarrow$ Better subtracting the gluon mass in the IR

Summary

- ERGE provides general framework for nonperturbative approximations.
- Quantitative behaviour of the propagators in the variational approach is partly reproduced.

- Outlook
  - Inclusion of Coulomb form factor.
  - Why RG-flow? Optimisation! Varying the regulator R_k.
  - Quarks