On the connection between Hamilton and Lagrange formalism in QFT

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Introduction

The Second Order Formalism

The path integral method within the Lagrange formalism

$$\langle 0_{\rm out} | 0_{\rm in} \rangle = \int \mathcal{D}q \exp\left[\frac{i}{\hbar}\int d^4x \mathfrak{L}(q, \nabla q, \dot{q})\right].$$

- It is a fundamental tool to formulate Quantum Field Theory.
- Symmetries and fundamental fields constitute aspects that are manifested explicitly.

The First Order Formalism

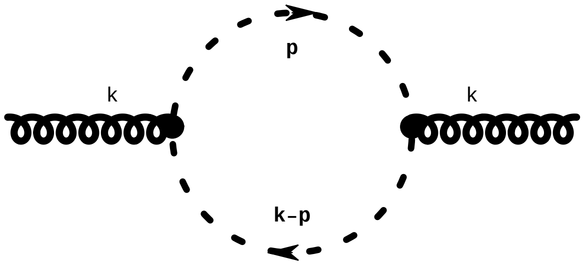
Within the canonical formalism is somewhat more cumbersome. Here the path integral looks like

$$\langle 0_{\text{out}}|0_{\text{in}}\rangle = \int \mathcal{D}q\mathcal{D}p \exp\left[\frac{i}{\hbar}\int d^4x \left(p\dot{q} - H(q, \nabla q, p)\right)\right].$$

- The “momentum fields” in such a case are not related to the fundamental fields by means of the canonical equations.
- The formal invariance properties of a generic theory admit additional transformation laws which are not present in the Lagrange formulation.

Advantages

This formalism result to be a successful tool to study the required complete cancelation of the energy divergences



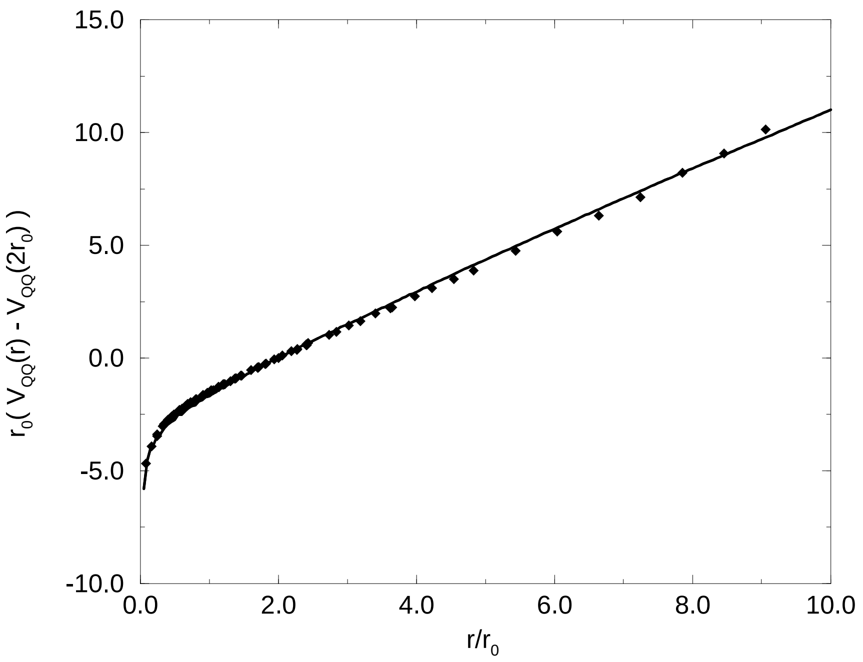
$$\sim \int \frac{dp_0}{2\pi} \int \frac{d^3\mathbf{p}}{(2\pi)^3} \frac{1}{(\mathbf{k}-\mathbf{p})^2\mathbf{k}^2} \to \infty$$

that emerge in a perturbative treatment of Coulomb gauge Yang-Mills theory within Lagrange framework.

- First order formalism is better suited for studying the renormalizability of Coulomb gauge Yang-Mills theory.

Confinement and outlook

The latter theory has attracted attention since one possible solution to the confinement problem in QCD is provided by the "Gribov-Zwanziger" scenario.



- Due to the dramatically complicated form of the functional equations in first order formalism an analysis in the second order formalism would be desirable.

Our Goal

- To provide general connections between the Greens functions of both formulations

This connection should help to perform the renormalization in the Langrange framework according to the insight in the renormalization procedure obtained in the Hamilton framework.

Hint

- At vanishing source associated to the momentum field the Effective Action and any correlation function involving just the fundamental field are the same in both formulations.

Schwinger-Dyson Equations

The Generating functionals

Let us suppose a vacuum-vacuum transition amplitude

$$\mathcal{Z}[J] = \langle 0_{\text{out}} | 0_{\text{in}} \rangle_J = \int \mathcal{D}[q]\mathcal{D}[p] \exp\left\{ \frac{i}{\hbar} I\left[q, p, J\right] \right\}.$$

Here the exponential argument has the structure

$$I = I_0\left[q, p\right] + \int d^4x J(x) \cdot \phi(x)$$

where

$$I_0\left[q, p\right] = \int d^4x \left\{ p_m(x)\dot{q}_m(x) - H\left(q, \nabla q, p\right) \right\}$$

whereas $\phi(x) \equiv (p_m(x), q_m(x))$ and $J(x) \equiv (J_{mp}(x), J_{mq}(x))$.

The Effective Action

The generating functional of the connected Feynman's graphs

$$\mathcal{W}^H[J] = -i \ln\left(\mathcal{Z}^H[J]\right), \quad \bar{\phi}(x) = \frac{\delta \mathcal{W}^H}{\delta J(x)}$$

Assuming that it is possible to invert these relations such that the sources are expressed as functionals of $\bar{\phi}$ we have

$$\Gamma^H[\phi] \stackrel{def}{\equiv} \mathcal{W}^H[J] - \int d^4x J(x)\cdot\bar{\phi}(x), \quad \frac{\delta \Gamma^H}{\delta \bar{\phi}(x)} = -J(x).$$

Deriving the DSEs

Our starting point is the generalization of the simple identity that the integral of a total derivative, under suitable boundary conditions, is zero

$$
\begin{aligned}
0 &= \int \mathcal{D}\phi \frac{\hbar}{i} \frac{\delta}{\delta \phi^i} \exp\left\{ \frac{i}{\hbar} \left(I_0[\phi] + J \cdot \phi \right) \right\} \\
&= \int \mathcal{D}\phi \left(\frac{\delta I_0[\phi]}{\delta \phi^i} + J_i \right) \exp\left\{ \frac{i}{\hbar} \left(I_0[\phi] + J \cdot \phi \right) \right\}
\end{aligned}
$$

- The substitution of each elementary objects by the derivative with regard to the respective classical sources allows to extract the integrand and to recognize the remaining part with $\mathcal{Z}[J]$.

Schwinger Dyson Equations

- . . .so that

$$\left.\frac{\delta I_0}{\delta\phi}\right|_{\phi(x)\to\frac{\hbar}{i}\frac{\delta}{\delta J(x)}} \mathcal{Z}[J] = -J(x)\mathcal{Z}[J].$$

- In term of $\mathcal{W}^H$

$$\left.\frac{\delta I_0}{\delta\phi}\right|_{\phi(x)\to\frac{\delta\mathcal{W}^H}{\delta J(x)}+\frac{\hbar}{i}\frac{\delta}{\delta J(x)}} = -J(x).$$

In the last step we made used the infinite dimensional version of the following identity

$$\mathrm{F}(\partial_\mathrm{x})\exp\{\mathrm{g}(\mathrm{x})\} = \exp\{\mathrm{g}(\mathrm{x})\}\mathrm{F}(\mathrm{g}'(\mathrm{x}) + \partial_\mathrm{x})$$

Matrix Structures

- By considering the definition of Γ^H

$$\frac{\delta \Gamma^H}{\delta \bar{\phi}(x)} = \frac{\delta I_0}{\delta \phi(x)} \bigg|_{\phi(x) \to \bar{\phi}[J](x) + \frac{\hbar}{i} \int d^4x' \mathbb{D}[J](x,x') \frac{\delta}{\delta \bar{\phi}(x')}}$$

The last equation involve the matrix

$$\mathbb{D}[J](x,x') \equiv \begin{pmatrix} \Delta^{pp}_{ml}[J](x,x') & \Delta^{pq}_{ml}[J](x,x') \\ \Delta^{qp}_{ml}[J](x,x') & \Delta^{qq}_{ml}[J](x,x') \end{pmatrix},$$

which is two-point connected Green's function.

Relation between Propagators

In correspondence the full proper propagator can be expressed as

$$\mathbb{G}[\phi](x'',x') \equiv \begin{pmatrix} \frac{\delta^2\Gamma^H}{\delta\bar{p_m}(x'')\delta\bar{p}_n(x')} & \frac{\delta^2\Gamma^H}{\delta\bar{p_m}(x'')\delta\bar{q}_n(x')} \\ \frac{\delta^2\Gamma^H}{\delta\bar{q_m}(x'')\delta\bar{p}_n(x')} & \frac{\delta^2\Gamma^H}{\delta\bar{q_m}(x'')\delta\bar{q}_n(x')} \end{pmatrix}$$

They are related via the identity

$$\int d^4x''\mathbb{D}[J](x,x'')\mathbb{G}[\phi](x'',x') = -\mathbb{I}.$$

- Both propagators are source-dependents.

Functional Connections

The Compact Notation

In what follows we will denote all the variables that a field can depend on by a single latin index.

- In case of a gauge field such indices will indicate the space-time point x, the vectorial index as well as the adjoint gauge group index a.
- Repeated indices are summed and integrated over for discrete and continuous variables, respectively.

Now let us consider

$$\Delta^{p\ldots p}_{i_1\ldots i_n} = \frac{\int \mathcal{D}[q]\mathcal{D}[p] p_{i_1} \ldots p_{i_n} \exp\left[\frac{i}{\hbar} I[q,p,J]\right]}{i^{1-n} \int \mathcal{D}[q]\mathcal{D}[p] \exp\left[\frac{i}{\hbar} I[q,p,J]\right]},$$

.

The Momentum Green's Function

We can express it as

$$\Delta^{p\ldots p}_{i_1\ldots i_n} = \frac{\int \mathcal{D}[p]\mathcal{D}[q] \frac{\hbar}{i}\frac{\delta}{J_{i_1 p}} \ldots \frac{\hbar}{i}\frac{\delta}{J_{i_n p}} \exp\left[\frac{i}{\hbar}\left\{I[q,p,J]\right\}\right]}{i^{1-n}\int \mathcal{D}[p]\mathcal{D}[q] \exp\left[\frac{i}{\hbar}\left\{I[q,p,J]\right\}\right]}$$

If the Hamiltonian is quadratic in the p, then we can perform the Gaussian functional integration over it, in whose case the integral in the exponential become in

$$\tilde{\mathcal{S}} = \mathcal{S} + \frac{1}{2}J_{ip}J_{ip} + J_{ip}\frac{\delta\mathcal{S}}{\delta\dot{q}_i} + J_{iq}q_i$$

where $\mathcal{S}$ really is the standard action.

Finding out $\Delta^{p\ldots p}_{i_1\ldots i_n}$

Through the functional methods it is possible to write

$$\Delta^{p\ldots p}_{i_1\ldots i_n} = \frac{\int \mathcal{D}[q] \left\{ \hat{\mathcal{O}}^{p\ldots p}_{i_1\ldots i_{n-1}} \left(J^p_{i_n} + \frac{\delta \mathcal{S}_0}{\delta \dot{q}_{i_n}} \right) \right\} \exp\left[\frac{i}{\hbar} \tilde{\mathcal{S}} \right]}{i^{1-n} \int \mathcal{D}[q] \exp\left[\frac{i}{\hbar} \tilde{\mathcal{S}} \right]}$$

where the operator

$$\hat{\mathcal{O}}^{p\ldots p}_{i_1\ldots i_{n-1}} = \prod_{l=1}^{n-1} \left(J^p_{i_l} + \frac{\delta \mathcal{S}_0}{\delta \dot{q}_{i_l}} + \frac{\hbar}{i} \frac{\delta}{\delta J^p_{i_l}} \right)$$

just acts on the function inside the curly brackets. Note that the integrand in the numerator is a functional dependent on the fields as well as the sources.

Master Equation for $\Delta^{p\ldots p}_{i_1\ldots i_n}$

The general expression for arbitrary momentum correlation functions,

$$\Delta^{p\ldots p}_{i_1\ldots i_n} = i^{n-1}\left[\hat{\mathcal{O}}^{p\ldots p}_{i_1\ldots i_{n-1}}\left(J^p_{i_n} + \frac{\delta\mathcal{S}_0}{\delta\dot{q}_{i_n}}\right)\Bigg|_{J^p=0}\right]_{q\to\frac{\delta\mathcal{W}}{\delta J^q}+\frac{\hbar}{i}\frac{\delta}{\delta J^q}}$$

- Here the connected Generating Functional in the standard Lagrange formalism appears which does not depend on J_p anymore.
- Any other m-point Green's function with n-external legs associated to the J^p's is obtained by taking $m-n$ derivatives with regard to J^q_m and setting them to zero.

The Averaged Momentum Field

By application of the latter equation we get the general expression for arbitrary momentum correlation functions,

$$\bar{p}_i[\bar{q}] = \left.\frac{\delta \mathcal{S}_0}{\delta \dot{q}_i}\right|_{q_m \to \bar{q}_i[J^q] + \frac{\hbar}{i}\Delta^{qq}_{ij}[J^q]\frac{\delta}{\delta q_j}} .$$

- The vacuum expectation value of $p-$field is generally not given by the usual definition of a canonical momentum field $\mathfrak{p}_i^{\text{can}}$ on the level of the Effective Action

$$\bar{p}_i[J=0] = \left\langle \frac{\delta S}{\delta \dot{q}} \right\rangle \neq \frac{\delta \Gamma}{\delta \dot{\bar{q}}} \equiv \mathfrak{p}_i^{\text{can}}$$

Δ_{ij}^{pp} and Δ_{ij}^{qp}

The $pp-$correlation functions can be expressed as

$$\Delta_{ij}^{pp} = \delta_{ij} + i \left(\frac{\delta \mathcal{S}_0}{\delta \dot{q}_i} \frac{\delta \mathcal{S}_0}{\delta \dot{q}_j} \right) \Bigg|_{q \to \bar{q}[J^q] + \frac{\hbar}{i} \Delta^{qq}[J^q] \frac{\delta}{\delta q}} ,$$

whereas the corresponding mixed $qp-$correlation function can be written as

$$\Delta_{ij}^{qp}[J^q] = \Delta_{il}^{qq} \frac{\delta}{\delta \bar{q}_l} \left(\frac{\delta \mathcal{S}_0}{\delta \dot{q}_j} \Bigg|_{q \to \bar{q}[J^q] + \frac{\hbar}{i} \Delta^{qq}[J^q] \frac{\delta}{\delta q}} \right) = \Delta_{il}^{qq} \frac{\delta \bar{p}_j}{\delta \bar{q}_l} .$$

- The other mixed Δ^{pq} is related by the bosonic nature of the fields $\Delta_{ij}^{pq} = \Delta_{ji}^{qp}$.

Inverting the Propagator $\mathbb{D}$

Once the individual elements of $\mathbb{D}-$propagator are computed the proper two-point function is completely determined in terms of the elements of the Lagrange framework. The inversion shows that

$$\mathbb{G} = \begin{pmatrix} \Gamma^{pp}_{ij} & \Gamma^{pp}_{il}\Delta^{pq}_{lm}\Gamma^{qq}_{mj} \\ \Gamma^{qq}_{il}\Delta^{qp}_{lm}\Gamma^{pp}_{mj} & {\Gamma^{qq}_{ij}}^{H} \end{pmatrix}$$

where $\Gamma^{pp}_{ij} = -\left(\Delta^{pp} + \Delta^{pq}\Gamma^{qq}\Delta^{qp}\right)^{-1}_{ij}$ and

$${\Gamma^{qq}_{ij}}^{H} = \Gamma^{qq}_{ij} + \Gamma^{qp}_{il}\left(\Gamma^{pp}_{lm}\right)^{-1}\Gamma^{pq}_{mj}.$$

A Generic Renormalizable Theory

The “Canonical Action”

In four dimensions the most general renormalizable “canonical action” can be expressed as a functional Taylor expansion,

$$
\begin{aligned}
I_0[q,p] &= I^{qp}_{0ji} p_i q_j + \frac{1}{2} I^{pp}_{0ij} p_i p_j + \frac{1}{2} I^{pqq}_{0ijk} p_i q_j q_k + \frac{1}{2} I^{qq}_{0ij} q_i q_j \\
&+ \frac{1}{3!} I^{qqq}_{0ijk} q_i q_j q_k + \frac{1}{4!} I^{qqqq}_{0ijkl} q_i q_j q_k q_l
\end{aligned}
$$

- Clearly, the coefficients $I^{\phi\ldots}_{0i\ldots}$ are field independent

$$
I^{\phi\phi}_{0ij\ldots} \equiv \frac{\delta}{\delta\phi_i} \frac{\delta}{\delta\phi_j} \frac{\delta}{\delta\phi_k} \ldots I_0 \Big|_{\phi=0} .
$$

The Action $\mathcal{S}$

In correspondence the most general four dimensional renormalizable action looks like

$$\mathcal{S} = \frac{1}{2}\mathcal{S}_{ij}q_i q_j + \frac{1}{3!}\mathcal{S}_{ijk}q_i q_j q_k + \frac{1}{4!}\mathcal{S}_{ijkl}q_i q_j q_k q_l .$$

In this context the following relations between the bare coefficients arise

$$\mathcal{S}_{ij} = I^{qp}_{0il} I^{pq}_{0lj} + I^{qq}_{0ij},$$

$$\mathcal{S}_{ijk} = I^{qqp}_{0ikn} I^{pq}_{0nj} + \bar{q}_k \leftrightarrow \bar{q}_j \text{permut.} + \bar{q}_i \leftrightarrow \bar{q}_j \text{permut.} + I^{qqq}_{0ijk},$$

$$\mathcal{S}_{ijkl} = I^{qqp}_{0ikn} I^{pqq}_{0njl} + \bar{q}_i \leftrightarrow \bar{q}_l \text{permut.} + \bar{q}_k \leftrightarrow \bar{q}_l \text{permut.} + I^{qqqq}_{0ijkl}.$$

The Canonical Momentum

To complete our analysis we point out that the general form of the quantum canonical momentum is given by

$$p_i^{can} = \mathcal{S}_{ji}^{q\dot{q}} q_j + \frac{1}{2} \mathcal{S}_{kji}^{qq\dot{q}} q_j q_k$$

Here

$$\mathcal{S}_{ij}^{q\dot{q}} = I_{0ij}^{qp} = \partial_{\tau_j} \delta_{ji} \quad \text{and} \quad \mathcal{S}_{ijk}^{qq\dot{q}} = I_{0ijk}^{qqp}.$$

- These polynomial representations allow to identify a priori the bare elements.
- The expansions given above prove to be very convenient to derive the DSEs for a general theory in both formulations

Diagrammatic Representations

We want to present the relations between correlation functions via explicit diagrammatic expressions.

I. The fundamental fields are represented by solid lines whereas the corresponding momentum fields are denoted by zigzag lines.

II. Dressed propagators are denoted by thick, whereas bare propagators and external lines by thin lines, respectively. Off-diagonal propagator components are represented by a thick, half solid and half zigzag line.

Additional Representations

III. The matrix propagator will be represented by a double line.

$\mathbb{D}$

IV. All proper correlation functions in the first and second order formalism are denoted by small and large filled blobs,

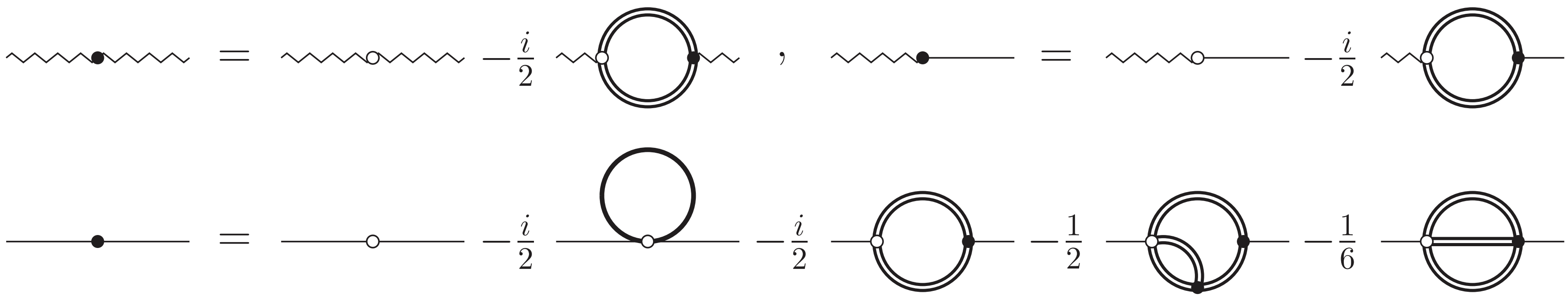
proper, Lagrange   proper, Hamilton

the bare vertex functions are represented by open blobs.

bare, Lagrange   bare, Hamilton

Ordinary Structure of DSEs

By considering the latter assignments, the ordinary diagrammatical representation of the coupled system of DSEs within the first order formalism is given by

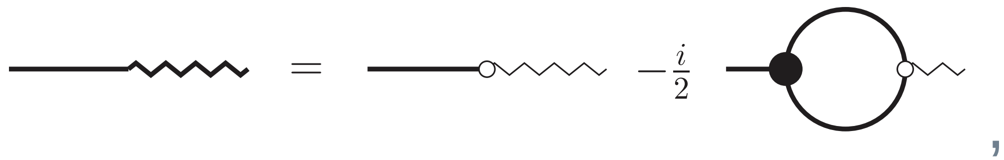


- The double lines represent the matrix propagator so that all possible graphs involving the individual propagators arise restricted by symmetries and the bare vertices of the theory.

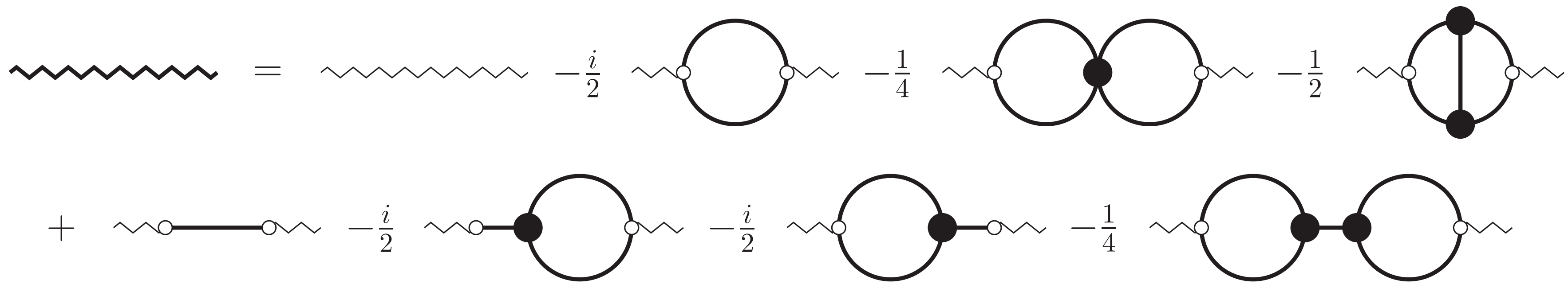
Diagrammatic Decompositions

Decomposition of the DSEs

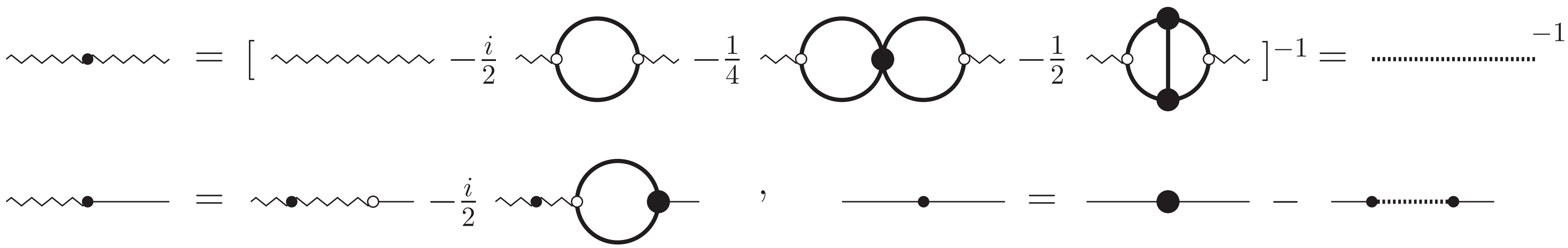
The mixed, connected $2-$point function looks like



whereas the pure momentum propagator



In correspondence the SDEs get the configuration



QED and q^4–Theory

For a theory without a three-point interaction vertex involving the time derivative of the fields the VEV of $\bar{p} = \dot{\bar{q}}$

$$\mathbb{G} = \begin{pmatrix} -\mathbb{I} & ik_0\mathbb{I} \\ ik_0\mathbb{I} & \Gamma^{qq} + k_0^2\mathbb{I} \end{pmatrix}$$

- Actually this result could be obtained from the standard procedure without setting the variables of our problem to zero.
- It means that only wave-function renormalization contributes to these kinetic terms.

Matching the $S-$matrix elements

To express proper functions in the Lagrange formalism in term of those of Hamilton framework we exploit the equivalence between them given by the stationarity in $\bar{p}$

$$\Gamma[\bar{q}] \equiv \Gamma^H\left[\bar{p}(\bar{q}), \bar{q}\right] \quad \text{whenever} \quad \frac{\delta \Gamma^H}{\delta \bar{p}} = -J^p = 0.$$

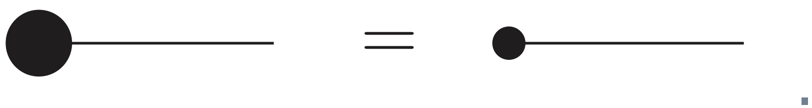
- Any proper n-point function in the standard formalism can be determined by taking n derivatives with respect to the fields in the above Equation and evaluated at the vacuum expectation value.

Deriving the Diagrammatic rules

Applying the chain rule, the first derivative reads

$$\frac{\delta \Gamma}{\delta \bar{q}_i} = \frac{\delta \Gamma^H}{\delta \bar{q}_i} + \frac{\delta \Gamma^H}{\delta \bar{p}_j} \frac{\delta \bar{p}_j}{\delta \bar{q}_i} = \frac{\delta \Gamma^H}{\delta \bar{q}_i}.$$

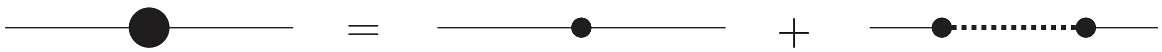
In our graphical representation this equation translates to



- As usual in the standard formalism, a differentiation with respect to a field is equivalent to attaching an external leg in the graphical representation.

First Diagrammatic Rule

Next we consider the second derivative whose graphical representation



give us the decomposition of the proper two-point Green's function.

- The fundamental field derivative of a proper correlation function in the Hamilton formalism yields the replacement rule

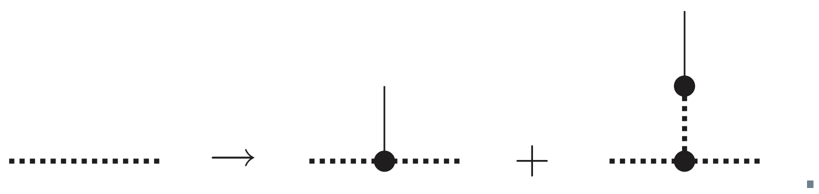
Here $\mathcal{D}^{pp} = -\left(\Gamma^{pp}\right)^{-1}$.

Second Diagrammatic Rule

- A field derivative can also act on the propagator $\mathcal{D}^{pp}$. Its derivative is obtained from the derivative of the inverse of an operator as

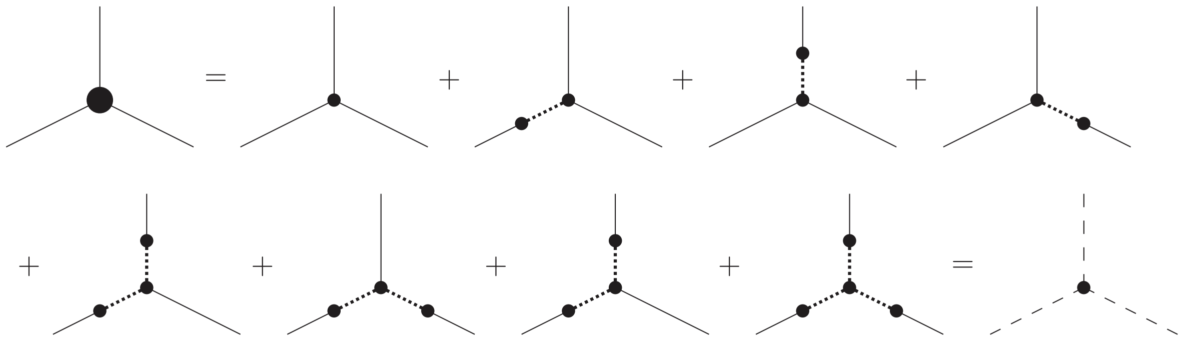
$$\frac{\delta \mathcal{D}^{pp}_{ij}}{\delta \bar{q}_k} = \mathcal{D}^{pp}_{im} \Gamma^{ppq}_{mnk} \mathcal{D}^{pp}_{nj}$$

which yields the graphical replacement rule



The $3-$point vertex

Applying these two replacement rules in all possible ways on the right hand side of the above equation for the two-point vertex yields immediately the corresponding decomposition of the proper 3-point vertex



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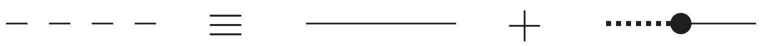


An Application

The results obtained here also apply to theories involving auxiliary fields. The functional integral to consider now is given by

$$\int D\psi^\dagger D\psi \exp\left(\frac{i}{\hbar}\int d^4x\left(\mathcal{L}_\psi + J_{\bar{\psi}}(x)\bar{\psi}(x) + J_\psi(x)\psi(x)\right)\right)$$

with the general Lagrangian of such a theory with local, quartic interactions

$$\mathcal{L}_\psi = \bar{\psi}(x)\left(i\partial\!\!\!/ - m_\psi\right)\psi(x) - \sum_i g^i_\psi\left(\bar{\psi}(x)\Gamma_i\psi(x)\right)^2$$

where the Γ_i are Dirac matrices.

Linearization of the Interaction

It can be performed by considering

$$\int D\psi^\dagger D\psi D\sigma \exp\left(\frac{i}{\hbar}\int d^4x\,\left(\mathcal{L}_\sigma\right.\right.$$
$$\left.\left. + J_{\bar\psi}(x)\bar\psi(x) + J_\psi(x)\psi(x) + J_\sigma^i(x)\sigma_i(x)\right)\right) \quad (1)$$

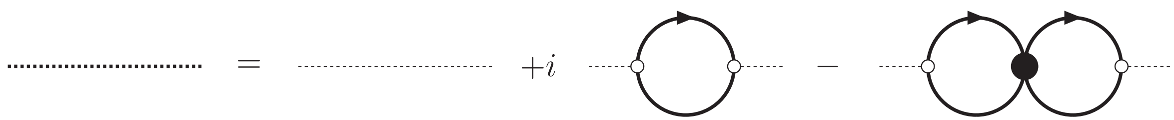
where we have introduced additional sources for the auxiliary fields σ_i and

$$\mathcal{L}_\sigma = \bar\psi(x)\left(i\partial\!\!\!/ - m_\psi + g_\sigma^i \Gamma_i \sigma_i(x)\right)\psi(x) + \frac{m_\sigma^2}{2}\sigma_i(x)^2$$

where $g_\psi = g_\sigma^2/(2m_\sigma^2)$.

The analogy

- Since the Lagrangian of the linearized theory is quadratic and lacks kinetic terms for the σ-fields, it can be integrated out retaining the sources for the auxiliary fields at this point.
- Due to the absence of 3-point interactions and mixing, the decomposition of the auxiliary σ-propagator is more simple



where the different prefactors of the loop correction arise due to the fermionic nature of the fields.

Summary

Summary

1. Given a quantum field theory in the context of the first order formalism it is possible to decompose all Green's functions in terms of those obtained from the second order formalism and vice versa.

2. We have discussed the connection between the Hamilton and the Lagrange formalism for a general quantum field theory and illustrated the detailed structure of the arising relations in the important case of a generic four-dimensional renormalizable field theory.

Summary

3 Although the structure of the DSE seems to be somewhat cumbersome, they are still more compact and simpler than the usual DSEs within the Hamilton formalism.

4 We finally showed, that the results obtained can also be applied to the case of theories involving auxiliary fields from a linearization of the interaction part of the action.

Thank You for Your Attention