The ’t Hooft-Loop in Coulomb Gauge

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Coulomb Gauge Yang-Mills Theory

- Hamiltonian formulation of Yang-Mills Theory

- Weyl gauge: $A_0 = 0$, D.O.F are cartesian coordinates $A_i^a(\mathbf{x})$, canonically conjugated momenta $\Pi_i^a(\mathbf{x}) = \frac{\delta}{i\delta A_i^a(\mathbf{x})}$,
  ETCRs $\left[A_i^a(\mathbf{x}), \Pi_j^b(\mathbf{y})\right] = i\delta^{ab}\delta_{ij}\delta(\mathbf{x}-\mathbf{y})$,
  Hamiltonian $\mathcal{H} = \int dx\, \Pi^2(x) + B^2(x)$

- Gauss' Law $\hat{D}_i \Pi_i |\mathbf{\Psi}\rangle = \rho_m |\mathbf{\Psi}\rangle$ guarantees invariance of the wave functional under small gauge transformations

- chose *Coulomb Gauge* $\partial_i A_i(\mathbf{x}) = 0$ to resolve Gauss' Law

Coulomb Gauge $\partial_i A_i(\mathbf{x}) = 0$

- eliminates unphysical degrees of freedom,
- change of coordinates from cartesian to curvilinear ones,
- introduces Jacobian (Fadeev-Popov determinant) $\mathcal{J}[A] = \det(-\partial_i D_i)$,
- new physical degrees of freedom are $A_i^{\perp a}(\mathbf{x})$.

Coulomb Gauge Yang-Mills Hamiltonian

$$\mathcal{H} = \frac{1}{2}\int d^3\mathbf{x}\left[\mathcal{J}^{-1}\Pi_i^a \mathcal{J}\Pi_i^a + B_i^a B_i^a\right] + \frac{g}{2}\int d^3\mathbf{x}\int d^3\mathbf{x}'\mathcal{J}^{-1}\rho^a(\mathbf{x})F^{ab}(\mathbf{x},\mathbf{x}')\mathcal{J}\rho^b(\mathbf{x}')$$

Physically motivated ansatz for the vacuum wave functional (strongly peaked at the Gribov Horizon $\mathcal{J} = 0$)

$$\mathbf{\Psi}[A] = \mathcal{J}^{-\frac{1}{2}}\mathcal{N}\exp\left(-\frac{1}{2}\int d^3\mathbf{x}d^3\mathbf{x}' A_i^{\perp a}\omega(\mathbf{x},\mathbf{x}')A_i^{\perp a}(\mathbf{x}')\right)$$

Find solutions for the Yang-Mills Schrödinger equation for the lowest energy eigenstate form the variational principle

$$\frac{\langle \mathbf{\Psi} | \mathcal{H} | \mathbf{\Psi} \rangle}{\langle \mathbf{\Psi} | \mathbf{\Psi} \rangle} \to \min .$$

$\rightsquigarrow$ coupled set of integral equations (*Dyson-Schwinger equations*), to be solved numerically[1].

[1] C. Feuchter and H. Reinhardt, Phys. Rev. **D70** (2004) 105021

Quantities which are computed self-consistently are

- ★ Gluon form factor $\omega(\mathbf{q})$,
  $\langle\omega|A_i^a(\mathbf{q})A_j^b(\mathbf{q}')|\omega\rangle = \frac{1}{2}(2\pi)^3 t_{ij}(\mathbf{q})\delta^{ab}\delta(\mathbf{q}-\mathbf{q}')\omega^{-1}(\mathbf{q})$

- ★ Ghost form factor $d(\mathbf{q})$, $G_\omega(\mathbf{q}) = G_0(\mathbf{q})d(\mathbf{q})/g$,
  $G_0(\mathbf{x},\mathbf{x}') = \langle\mathbf{x}|(-\partial^2)^{-1}|\mathbf{x}'\rangle = \frac{1}{4\pi|\mathbf{x}-\mathbf{x}'|}$

- ★ Curvature $\chi(\mathbf{q})$, represents the ghost-loop part of the gluon self-energy,
  $\chi = -\frac{1}{2}\langle\frac{\delta^2 \ln \mathsf{det}(-D\partial)}{\delta A \delta A}\rangle$,
  $\mathcal{J}[A] = \mathsf{det}(-D\partial) = \exp(-\int A\chi A)$

The following IR behaviour holds to leading order:

$$\omega(q) = \chi(q) = A \cdot k^{-\alpha}, \quad d(q) = B \cdot k^{-\beta}, \quad 2\beta - \alpha = 1$$

The existence of two kinds of solutions has been predicted:

★ One with $\alpha \approx 0.795$ (Found by C. Feuchter 2004)

★ One with $\alpha = 1$

We have been able to obtain also the solution with $\alpha = 1$. Very careful and elaborate numerics is required for it.

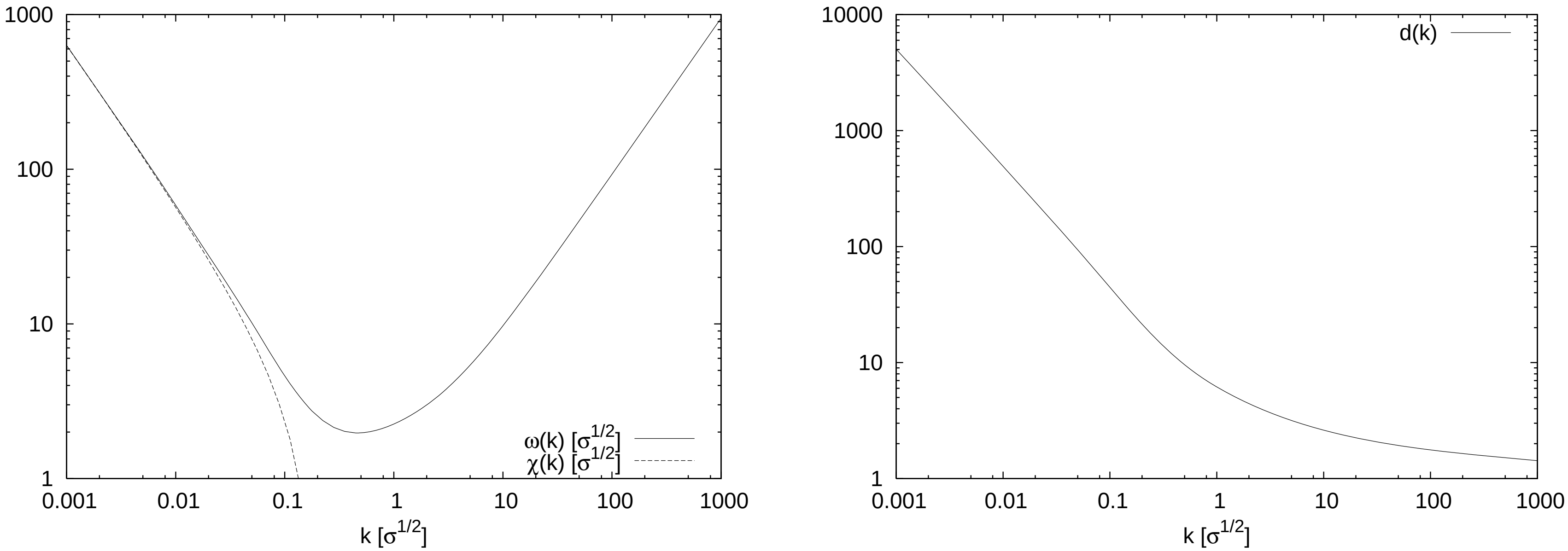


Figure 1: (left) The Gluon dressing function $\omega(q)$ and the curvature $\chi(q)$, calculated by minimizing the vacuum energy functional with a gaussian ansatz for the wave functional. These curves are from the $\alpha = 1$ solution. (right) Ghost dressing function $d(q)$ from the same calculation.

Phys.Rev.**D75**:045011,2007; `hep-th/0612241`

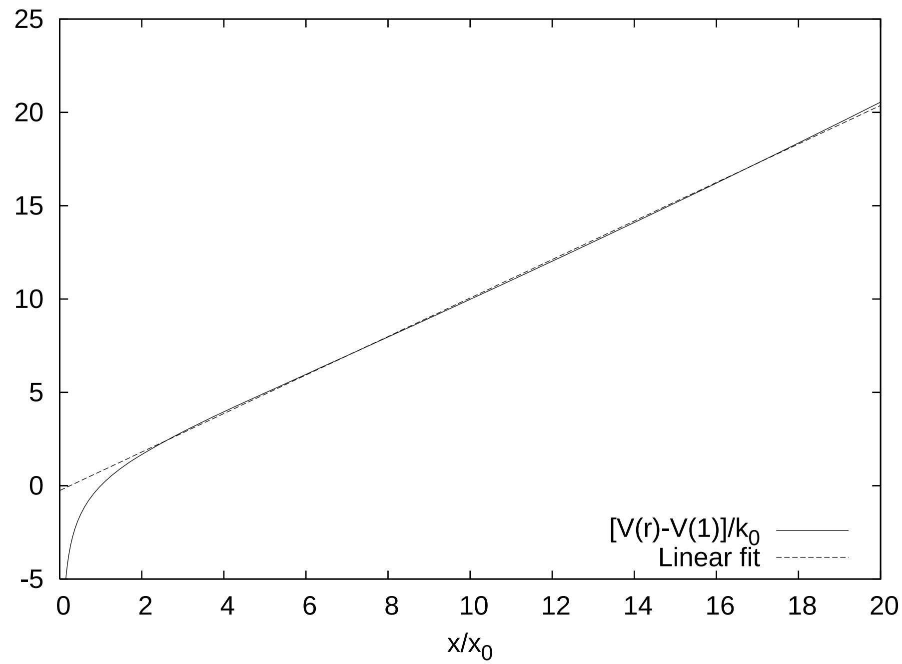


Figure 2: The Coulomb Potential (quark-antiquark-Potential), computed with the form factors from the solution with $\alpha = 1$. We find a linear rising potential within excellent precision (numerical error $< 1\%$).

The 't Hooft Loop as confinement criterion

The Wilson loop $W[A(\mathcal{C})] = \mathrm{tr}\,\mathrm{P}\,\exp(-\oint dx_\mu A^\mu)$

★ Gauge-invariant quantity

★ Measures magnetic flux through surface (for abelian theories)

Temporal Wilson Loop: Order parameter for Yang-Mills Theory

$$\langle W[A(\mathcal{C})]\rangle = \begin{cases} \exp(-\sigma \cdot \mathrm{Area}(\mathcal{C})), & \text{confined phase} \\ \exp(-\kappa \cdot \mathrm{Perimeter}(\mathcal{C})), & \text{deconfined phase} \end{cases}$$

Spatial 't Hooft loop $V(\mathcal{C})$: defined via spatial Wilson Loop

$$V(\mathcal{C}_1)W(\mathcal{C}_2) = Z^{L(\mathcal{C}_1,\mathcal{C}_2)}W(\mathcal{C}_2)V(\mathcal{C}_1)$$

with $Z = -1$ for $SU(2)$ and L = Gaussian linking number.

't Hooft loop behaviour:

$$\langle V[A](\mathcal{C})\rangle = \begin{cases} \exp(-\tilde{\kappa} \cdot \text{Perimeter}(\mathcal{C})), & \text{confined phase} \\ \exp(-\tilde{\sigma} \cdot \text{Area}(\mathcal{C})), & \text{deconfined phase} \end{cases}$$

$\rightsquigarrow$ dual to the Wilson loop's behaviour.

(G. 't Hooft, Nucl. Phys. **B138** (1978) 1) 't Hooft loop creates a Center Vortex ($\mathbf{A}(\mathcal{C})$ denotes a center vortex field, $W[\mathbf{A}(\mathcal{C}_1)](\mathcal{C}_2) = Z^{L(\mathcal{C}_1,\mathcal{C}_2)}$):

$$V(\mathcal{C})\Psi[A] = \Psi(A + \mathbf{A})$$

Continuum Representation:

(H. Reinhardt, Phys. Lett. **B577** (2003) 317-323)

$$V(\mathcal{C}) = \exp\left[i \int d^3x \mathbf{A}(\mathcal{C})(x)\Pi(x)\right]$$

The 't Hooft Loop in Coulomb Gauge

To one-loop order, one can calculate

$$\begin{aligned}\langle V(\mathcal{C})\rangle &= \int \mathcal{D}A \,\mathsf{det}(-D\partial)\Psi^*(A)\Psi(A+\mathbf{A}) \\ &= \exp(-S(\mathcal{C})), \quad S(\mathcal{C}) := S(R)\end{aligned}$$

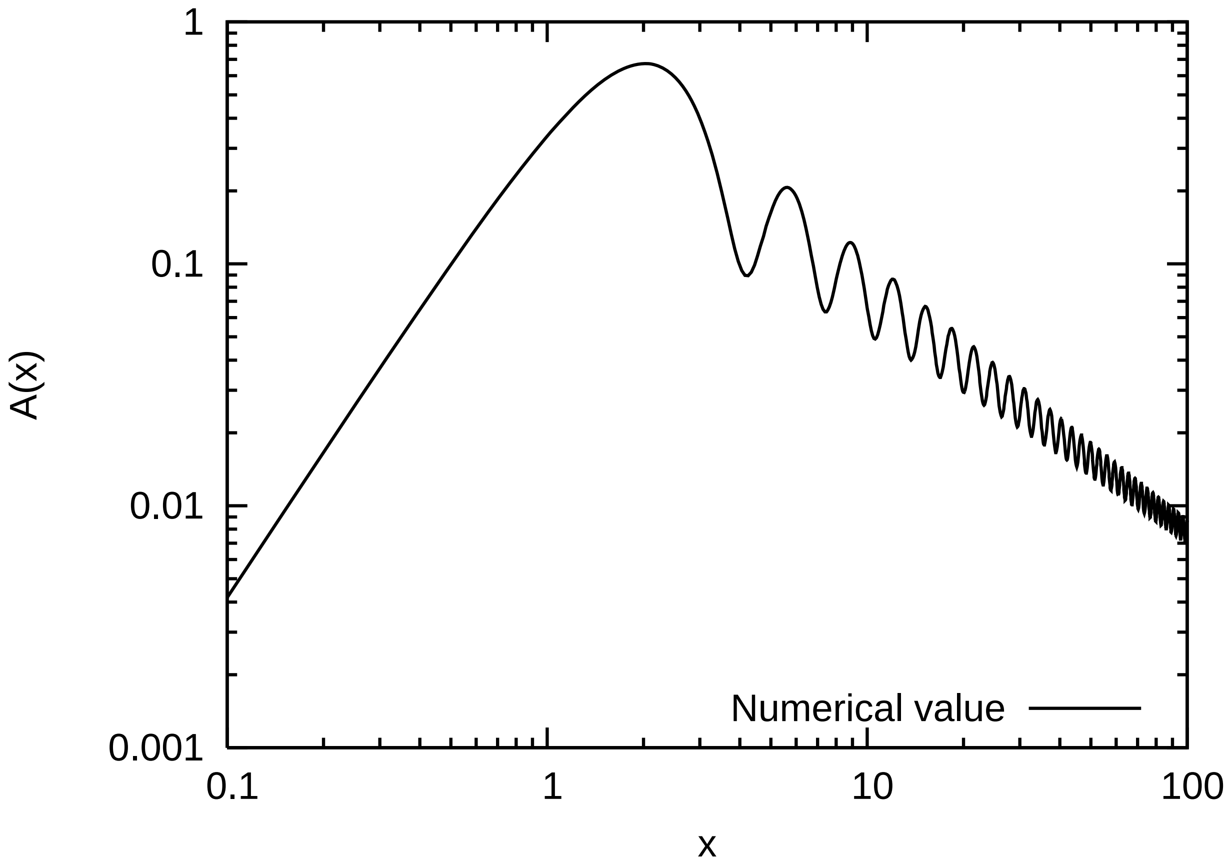
$$S(R) = 4\pi R \int_0^\infty dx \, \underbrace{K\left(\frac{x}{R}\right)}_{\text{physics}} \underbrace{\int_0^{\pi/2} d\alpha \, (1 - 2\sin^2\alpha)\, f(2x\sin\alpha)}_{=:A(x),\,\text{geometry}}$$

where $f(z) = j_0(z) - j_0^{''}(z)$, $j_0(z) = \frac{\sin z}{z}$.

The geometrical integral $A(x)$

$$
\begin{aligned}
A(x) &= \int_0^{\pi/2} d\alpha \, (1 - 2\sin^2\alpha) \, f(2x\sin\alpha) \\
&= \frac{\pi}{4} \int_{-1}^{1} dz (1 + z^2) J_2(2xz) \\
&= \frac{\pi}{4x} \Big[2x J_0(2x) + \pi x \big(J_1(2x) H_0(2x) - J_0(2x) H_1(2x) \big) - 2 J_1(2x) \Big] \\
&\quad - \frac{3\pi^2}{16x^2} \Big[J_2(2x) H_1(2x) - J_1(2x) H_2(2x) \Big]
\end{aligned}
$$

The geometrical integral $A(x)$

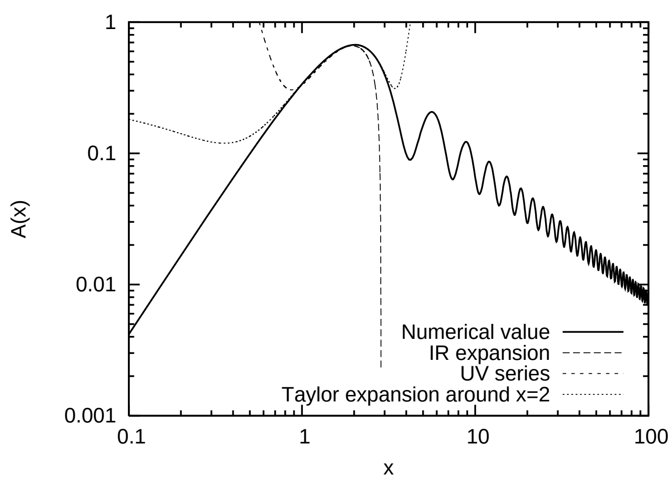


The geometrical integral $A(x)$

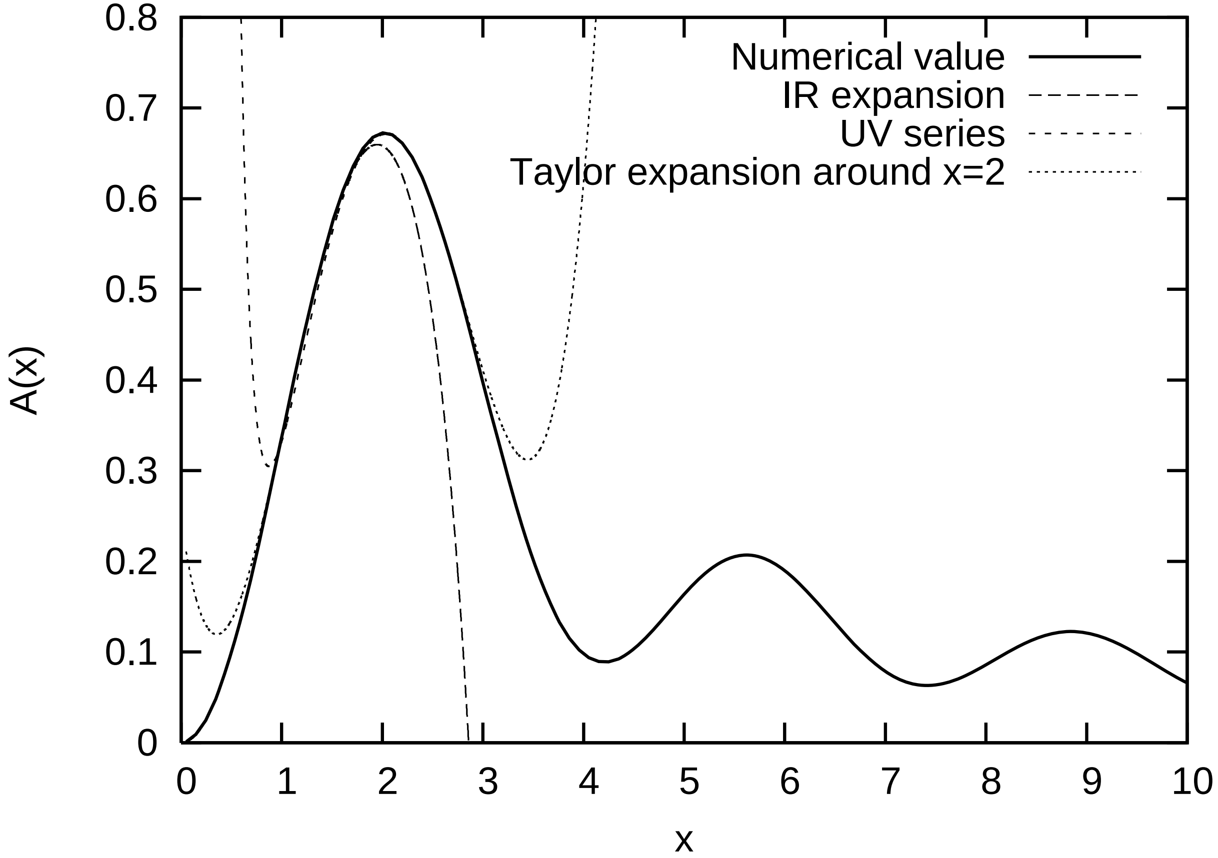
Asymptotical properties:

★ $x \to 0$: $A(x) = \frac{2\pi}{15}x^2 - \frac{\pi}{35}x^4 + O\left(x^6\right)$

★ $x \to \infty$: $A(x) = \frac{\pi}{4x} + \frac{\sqrt{\pi}}{2}\cos\left(2x + \frac{\pi}{4}\right)\frac{1}{\sqrt{x^3}} + O\left(\frac{1}{\sqrt{x^5}}\right)$



The geometrical integral $A(x)$



Physics: $K(q)$

$$K(q) = [\omega(q) - \chi(q)] \left[1 - \frac{1}{2} \frac{[\omega(q) - \chi(q)]}{\omega(q)} \right]$$
$$= \frac{1}{2}\omega(q) \left[1 - \left(\frac{\chi(q)}{\omega(q)} \right)^2 \right]$$

Asymptotic behavior:

★ IR: $K(q) \stackrel{\text{IR}}{=} [\omega(q) - \chi(q)] \approx c_0 + c_1\, q + \ldots$

★ UV: $K(q) \stackrel{\text{UV}}{=} \frac{1}{2}\omega(q) \approx \frac{1}{2}(q + a_0) + \ldots$

Discussion of $S(R)/R$

$$S(R) = 4\pi R \int dx\, K\left(\frac{x}{R}\right) A(x)$$

★ Integral for $S(R)$ linearly and logarithmically UV-divergent

★ Large-R-behavior of $S(R)$ tests IR-behavior of $K(q)$

$\rightsquigarrow$ need to renormalize by subtraction of UV-leading parts in $K(q)$

Crucial: do not spoil IR-behavior!

Parametrize $\omega(q) = A/q + a_0 + q$.
Replace $K(q) \rightarrow K(q) - K_0(q) = \bar{K}(q)$:

$$K(q) = \frac{1}{2}\omega(q)\left[1 - \left(\frac{\chi(q)}{\omega(q)}\right)^2\right]$$

$$K_0(q) = \frac{1}{2}(q + a_0)\left[1 - \left(\frac{\chi(q)}{\omega(q)}\right)^2\right]$$

$$\boxed{\bar{K}(q) = \frac{A}{q}\left[1 - \left(\frac{\chi(q)}{\omega(q)}\right)^2\right]}$$

This subtraction has the following properties:

★ We do not spoil the IR behavior

★ Subtract all terms creating UV divergencies in $S(R)$

★ Subtract also terms creating UV-finite contributions to $S(R)$, however, these are contributions are **neglegible**

$\rightsquigarrow$ Renormalized 't Hooft loop exponent $\bar{S}(R)$.

$$\bar{S}(R) = 4\pi R \int dx \, \frac{A}{q} \left[1 - \left(\frac{\chi(q)}{\omega(q)} \right)^2 \right] A(x), \qquad q = x/R$$

Behaviour of the ’t Hooft Loop for different solutions of the YM vacuum

Using the asymptotic behavior of $A(x)$, one can show that:

- ★ A finite c_0 leads to a logarithmically diverging $\bar{S}(R)/R$
- ★ $c_0 = 0$ leads to $\bar{S}(R \to \infty)/R \to \mathrm{const.}$, which corresponds to a perimeter law for the ’t Hooft Loop.

These points can be verified numerically also with the full $A(x)$.

Result for the 't Hooft Loop

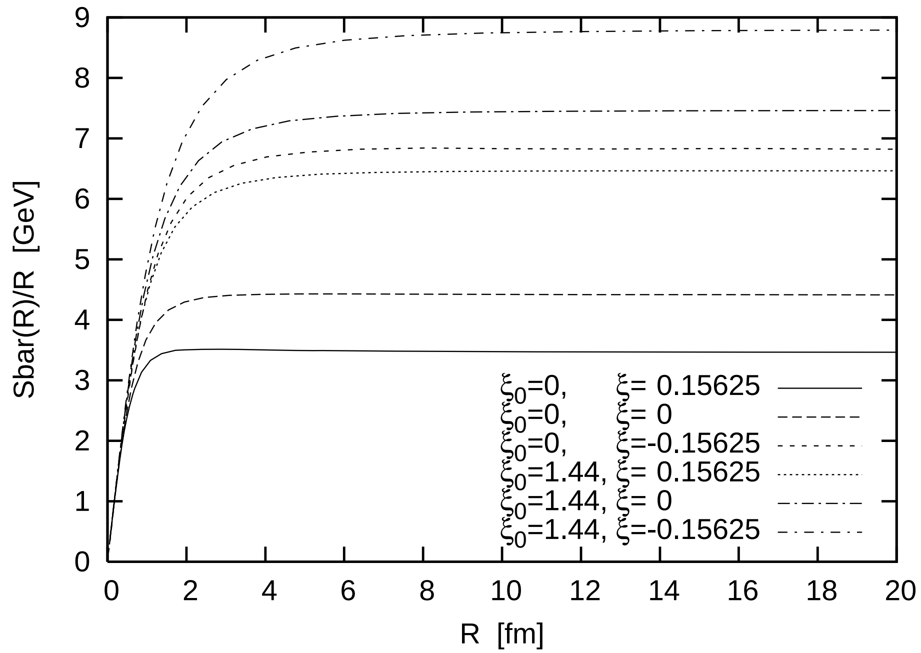


Figure 3: The $\bar{S}(R)/R$ function, calculated numerically using the full expression for $A(x)$ and the renormalized $\bar{K}(q)$ as described in the text. The YM-Solution has the property $\omega(q) - \chi(q) \longrightarrow 0$ for $q \longrightarrow 0$. We see a behaviour of $\bar{S}(R \longrightarrow \infty)/R = \mathrm{const.}$, which means a perimeter law for the 't Hooft loop, which corresponds to a confined theory.

Summary

★ We have been able to obtain solutions for the Yang-Mills vacuum which show a exact linear Coulomb potential.

★ We can show a perimeter-law for the 't Hooft Loop, i.e. confinement, for a special solutions of the Yang-Mills vacuum.