Adiabatic regularization of infrared divergences in a scalar Quantum Field Theory

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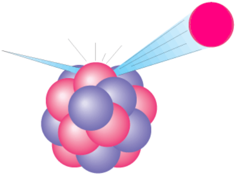
Motivation

- This topic was the subject of my master thesis.
- It was the aim to get an understanding of QFT (Quantum Field Theory)
- There are different methods to regularize the QFT. The method I choosed is the one, that uses test functions g(x).

Scattering of a particle on an electromagnetic potential (e.g. an atomic nucleus)

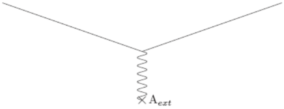


The measured differential cross-section is finite! $\Rightarrow \frac{d\sigma}{d\Omega} < \infty$

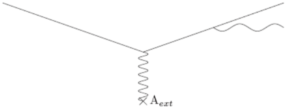
The experimental result should be in accordance with the theory. Calculations should yield a finite cross-section, too!

In theory, there are two processes of tree level we must take into account:

- The first process is the pure scattering process.



- The second process is the scattering process, where the particle emits a soft photon.



If $E_\gamma < E_0$ (experimental energy-threshold), the photon cannot be detected. The two processes are indistinguishable.

Definition of the theory

We treat the following problem in a scalar QFT

All particles are treated as spinless: $\mathcal{L} = \mathcal{L}_{KG}^{m} + \mathcal{L}_{KG}^{0} + \mathcal{L}_{int}$

- $\mathcal{L}_{KG}^{m} = (\partial_\mu \phi^\dagger)(\partial^\mu \phi) - m^2 \phi^\dagger \phi$ (massive scalar field)
- $\mathcal{L}_{KG}^{0} = (\partial_\mu A)(\partial^\mu A)$ (massless scalar field)
- $\mathcal{L}_{int} = -e : \phi \phi^\dagger A :$ (interaction)

where e is the coupling constant. (with dimension of an energy)

We are interested in $\mathcal{H}_{int}$. $\Rightarrow$ Legendre transformation of $\mathcal{L}$.

- $\mathcal{H}_{int} = e : \phi \phi^\dagger A :$ (interaction Hamiltonian density)

The field operators are defined as follows:

- $\phi(x) = \frac{1}{(2\pi)^{(3/2)}} \int \frac{d^3k}{\sqrt{2k_0}} [a(\vec{k})e^{-ikx} + b^\dagger(\vec{k})e^{ikx}]$
- $\phi^\dagger(x) = \frac{1}{(2\pi)^{(3/2)}} \int \frac{d^3k}{\sqrt{2k_0}} [a^\dagger(\vec{k})e^{ikx} + b(\vec{k})e^{-ikx}]$
- $A(x) = \frac{1}{(2\pi)^{(3/2)}} \int \frac{d^3k}{\sqrt{2k_0}} [c(\vec{k})e^{-ikx} + c^\dagger(\vec{k})e^{ikx}] + A_{ext}(x)$

where a,b,c and $a^\dagger, b^\dagger, c^\dagger$ are annihilation and creation operators.

They are defindеd as follows:

- $a^\dagger(\vec{k})|n_{\vec{k}}\rangle = \sqrt{n_{\vec{k}} + 1}|n_{\vec{k}} + 1\rangle$
- $a(\vec{k})|n_{\vec{k}}\rangle = \sqrt{n_{\vec{k}}}|n_{\vec{k}} - 1\rangle$

An essential quantity in QFT is the scattering matrix $\hat{S}$.
It is defined as follows:

- $\hat{S} := U(\infty, -\infty)$

But how does $\hat{S}$ look like?
We consider the Schroedinger equation of a state.

- $|\psi(t)\rangle = U(t, t_0)|\psi(t_0)\rangle$
- $i\frac{d}{dt}|\psi(t)\rangle = H_{int}(t)|\psi(t)\rangle$

We obtain the following diff. eq. for U:

- $i\frac{d}{dt}U(t, t_0) = H_{int}(t)U(t, t_0)$

Since $U(t_0, t_0) = \mathbb{1}$, we can rewrite the equation

- $U(t, t_0) = \mathbb{1} - i \int\limits_{t_0}^{t} dt_1 H_{int}(t_1) U(t_1, t_0)$

By iteration, we get a series.

- $U(t, t_0) = \mathbb{1} + \sum\limits_{n=1}^{\infty} \frac{(-i)^n}{n!} \int\limits_{t_0}^{t} dt_1 ... \int\limits_{t_0}^{t} dt_n T(H_{int}(t_1) ... H_{int}(t_n))$

Finally, we obtain

- $\hat{S} = \mathbb{1} + \sum_{n=1}^{\infty} \hat{S}^{(n)}$ with

  $\hat{S}^{(n)} = \frac{(-i)^n}{n!} \int d^4x_1 ... d^4x_n \; T(\mathcal{H}_{int}(x_1) ... \mathcal{H}_{int}(x_n))$ where

  $H_{int}(t) = \int d^3\vec{x} \mathcal{H}_{int}(x^\mu)$

T $=$ Time ordering operator! The $\mathcal{H}_{int}$'s contain all physical information!

But there is a problem with $\hat{S}$: $\hat{S}$ is not mathematically well defined!!!

Delta distributions just make sense in an integral, e.g.

- $\int \delta(x-a)g(x)dx = \langle \delta_a | g \rangle = g(a)$

when g(x) is a test function. (g $\in$ S)
Test functions are smooth and rapidly decreasing functions.

The space of these functions is called Schwartz-Space.

- $S(\mathbb{R}) = \{g \in C^\infty(\mathbb{R}) \big| \, ||g||_{\alpha,\beta} < \infty \; \forall \alpha, \beta\}$ *with*

  $||g||_{\alpha,\beta} = ||x^\alpha \frac{d^\beta}{dx^\beta} g(x)||_\infty$

For the scattering matrix, we obtain

- $\hat{S}(g) = \mathbb{1} + \sum_{n=1}^{\infty} \frac{1}{n!} T_n(x_1, \ldots, x_n) g(x_1) \cdot \ldots \cdot g(x_n)$

$g(x_1, \ldots, x_n)$ $(= g(x_1) \cdot \ldots \cdot g(x_n))$ constricts the interaction into a certain space-time volume.

$\hat{S}$ is now mathematically well defined, but unphysical!

$\longrightarrow$ Introduction of a scaling parameter ϵ. We define

- $g(x_1, \ldots, x_n) = g_0(\epsilon x_1, \ldots, \epsilon x_n) \quad with \quad g_0(0, \ldots, 0) = 1$

If $\epsilon \longrightarrow 0$, then the interaction is extended to infinity.

Finally, we are interested in the cross-section:

- $\frac{d\sigma}{d\Omega} \sim |S_{fi}|^2 \qquad$ with $S_{fi} := \langle f|\hat{S}|i\rangle$

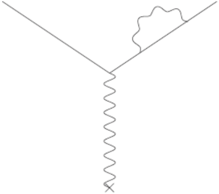
For the scattering process with soft photon emission, the Feynman graph is of second order in the matrix. (tree graph)

- $S_{fi_{2^{nd}\,order}} \Rightarrow \frac{d\sigma}{d\Omega}_{4^{th}\,order}$

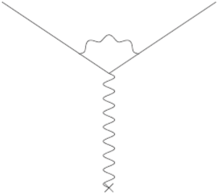
For the pure scattering process, we need all Feynman graphs, that contribute to the cross-section of 4^{th} order!!!

- $\frac{d\sigma}{d\Omega}_{4^{th}\,order} \sim |S_{fi_{2^{nd}\,order}}|^2 + 2\mathbb{Re}(S^*_{fi_{1^{st}\,order}} S_{fi_{3^{rd}\,order}})$

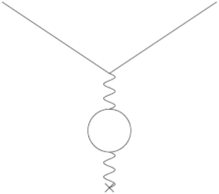
The Feynman diagrams at 3^{rd} order



self-energy graph



vertex graph



vacuum polarization

The last graph (vacuum polarization) is infrared finite.

The contribution to the scattering process where a soft photon is emitted only from a tree graph:

- $(\frac{d\sigma}{d\Omega})^{B}_{div} = \frac{d\sigma^1}{d\Omega}\frac{e^2}{2(2\pi)^2 m^2}\left[-1 + \frac{1}{b\sqrt{1+b^2}}\log\left|b+\sqrt{1+b^2}\right|\right]\log|\epsilon|$

where $\frac{d\sigma^1}{d\Omega}$ is the 1^{st} order cross-section and $b = \frac{|\vec{P}|}{2m}$ with $\vec{P} = \vec{p}_f - \vec{p}_i$

Contributions to the pure scattering process:

- $(\frac{d\sigma}{d\Omega})^{S}_{div} = \frac{d\sigma^1}{d\Omega}\frac{e^2}{2(2\pi)^2 m^2}\log|\epsilon|$
- $(\frac{d\sigma}{d\Omega})^{V}_{div} = \frac{d\sigma^1}{d\Omega}\frac{e^2}{2(2\pi)^2 m^2}\left[-\frac{1}{b\sqrt{1+b^2}}\log\left|b+\sqrt{1+b^2}\right|\right]\log|\epsilon|$

Conclusion

1. The adiabatic limit exists!!!

$$\Rightarrow (\frac{d\sigma}{d\Omega})^{B}_{div} + (\frac{d\sigma}{d\Omega})^{S}_{div} + (\frac{d\sigma}{d\Omega})^{V}_{div} = 0$$

2. The infrared problem has been solved in a strictly mathematical, perturbative scheme.

3. The method of Epstein-Glaser (with test functions g(x)) is an alternative to the method, where a finite photon mass is introduced.