Computational Hydrodynamics

Introduction to basic Methods

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UNI
BASEL

Outline

Why hydrodynamical description

- A large class of astrophysical problems involve collisional systems where the mean free path is much smaller than all lentgh scales of interest.
  $\Rightarrow$ Adopt a fluid description of matter
- Simplest case: single, ideal, non-magnetic fluid and no external forces

Euler equations I

Set of conservation laws for

- Mass:

$$\frac{\partial \rho}{\partial t} + \frac{\partial \rho v_i}{\partial x_i} = 0$$

- Momentum:

$$\frac{\partial(\rho v_i)}{\partial t} + \frac{\partial}{\partial x_j}\left(\rho v_i v_j + p\delta_{ij}\right) = 0$$

- Energy:

$$\frac{\partial E}{\partial t} + \frac{\partial}{\partial x_j}\left[(E + p)\, v_j\right] = 0$$

(Total) Energy

$$E = \frac{\rho}{2} v_j v_j + e$$

Non-linear hyperbolic system of PDEs

Euler equations II

- The Euler equation are incomplete
  $\Rightarrow$ Equation of state required to close system

$$p = p(\rho, T), e = e(\rho, T)$$

- Integral vs differential form of the Euler equations:

$$\frac{\partial \mathbf{u}}{\partial t} + \frac{\partial \mathbf{F}_j}{\partial x_j} = 0$$

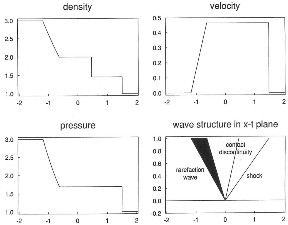
$$\int_V \frac{\partial \mathbf{u}}{\partial t} \mathrm{d}V + \int_{\partial V} \mathbf{F}_j \cdot \mathrm{d}\mathbf{S} = 0$$

where $\mathbf{u}$ vector of conserved variables and $\mathbf{F_j}(\mathbf{u})$ flux vector

Temporal change in conserved variables in volume V equals to gains and losses through boundary of volume ∂V

Euler equations III

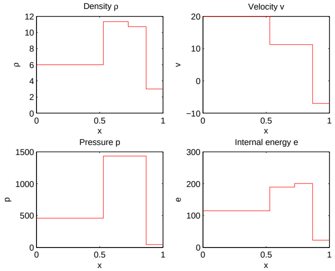
- Euler equations allow discontinuous solutions
  Shock tube



- Certain condition for stability of discontinuities:
  Rankine-Hugoniot jump conditions
- Derivatives ill defined at discontinuities
  $\Rightarrow$ Have to use integral form of the Euler eqs

Euler equations IV

- Riemann problem
- Not restricted to zero initial velocity
- Analytical solution exits
- Requires solving a nonlinear equation $\Rightarrow$ Newton iterations
- Solution is self-similar and discontinuites separated by constant flow



Discretization I

The Euler equations are a non-linear system of 1st order PDEs

- Numerical methods replace continuous problem represented by the PDEs by a finite set of discrete values
- Discretize time into discrete steps
  - Implicit method
  - Explicit method
- Discretize space into finite set of points or volumes
  - Finite differences methods
  - Finite volume methods

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Focuss in the following on explicit finite volume methods

Discretization II

Explicit finite volume metods

- Discretize space into finite volumes or cells and define averages

$$\mathbf{u}^n_{i,j,k} = \frac{1}{V}\int_V \mathbf{u}(\mathbf{x},t)\mathrm{d}V \ , \ V = \Delta x_i \Delta y_j \Delta z_k$$

$\Rightarrow$ Piece-wise constant distribution over grid

- Time step restricted by CFL condition for stability (Courant, Friedrichs, Lewy)

$$\Delta t < \Delta t_{CFL} = \min_i \left(\frac{\Delta x_i}{c_{max}}\right) \ , \ c_{max}\text{: maximum characteristic speed}$$

i.e. information must propagate by at most one zone per time step

High resolution methods I

Desirable & necessary properties of high resolution shock capturing methods (HRSC)

- Automatic, stable and sharp resolution of flow discontinuities (without execessive smearing)
- Consistency with the Euler eqs: convergence under grid refinement to physically correct solution
- High-order accuracy

High resolution methods II

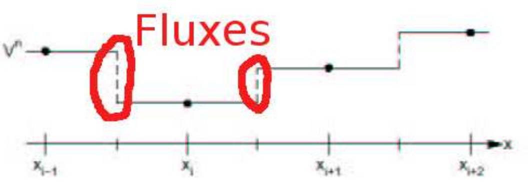
Automatic, stable and sharp resolution of flow discontinuities (without execessive smearing)

- Have to use conserved variables to get correct jump conditions for shocks
- Jump conditions derived from integral formulation of Euler eqs (Rankine-Hugoniot jump conditions)
- Have to use conservative numerical methods

$$u_i^{n+1} = u_i^n + \frac{\Delta t}{\Delta x}\left[f_{i+1/2} - f_{i-1/2}\right]$$

High resolution methods II

Automatic, stable and sharp resolution of flow discontinuities (without exececessive smearing)



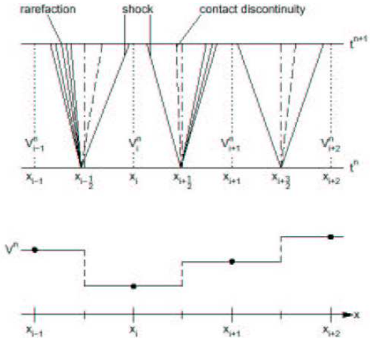
- Have to use conservative numerical methods

$$u_i^{n+1} = u_i^n + \frac{\Delta t}{\Delta x}\left[f_{i+1/2} - f_{i-1/2}\right]$$

Not obvious to find flux at cell boundaries with piece-wise constant data

Godunov's method I

Conservative extension of first order upwind scheme to nonlinear equations



$$u_i^{n+1} = u_i^n + \frac{\Delta t}{\Delta x}\left[f_{i+1/2} - f_{i-1/2}\right]$$

Numerical flux is obtained by solution of local Riemann problems

$$f_{i+1/2} = \frac{1}{\Delta t}\int_{t^n}^{t^n+\Delta t} f(u(x_{i+1/2}, t))\mathrm{d}t$$

Godunov's method I

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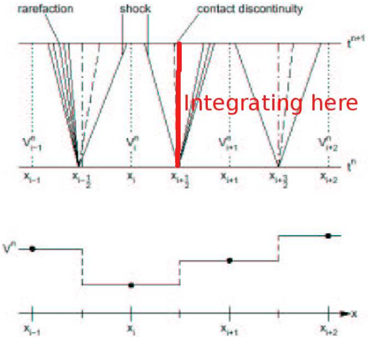

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Godunov's method II

Godunov's method is:

- Stable (under CFL condition)
- Converges to physical solution
- Shock capturing

But

- Only first order accurate
- Very diffusive
- Solving a Riemann problem is expensive

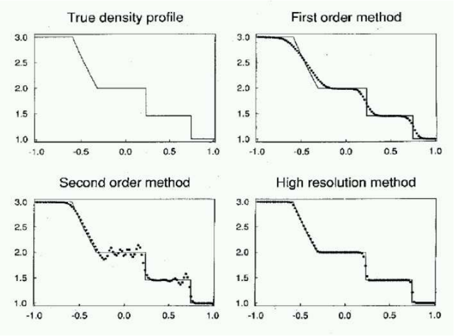

Figure: Figure taken from LeVeque

Godunov's method III

Improving Godunov's method:

- Use approximate Riemann solvers
- Avoid expensive iteration
- Prominent: Roe, HLL(E)

Godunov's method III

Improving Godunov's method:

- Use approximate Riemann solvers
- Use piece-wise linear representation

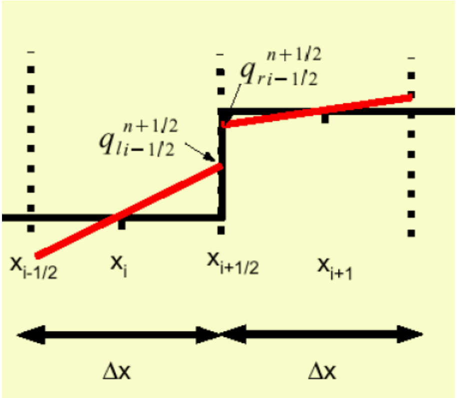

Figure: Figure taken from E. Müller

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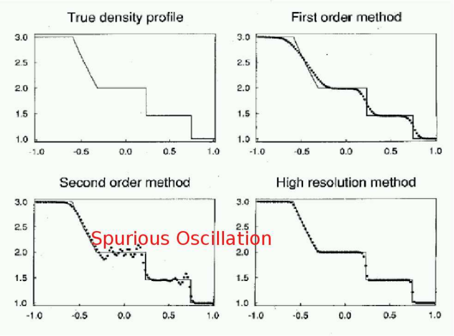

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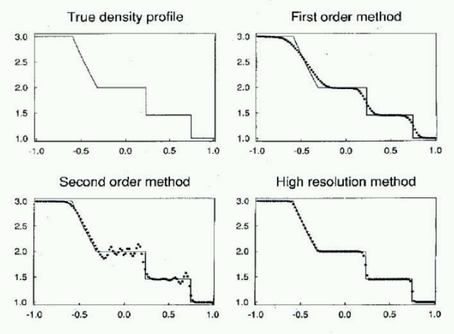
Godunov's method III

Improving Godunov's method:

- Use approximate Riemann solvers
- Use piece-wise linear representation
- Nonlinear stability condition $\Rightarrow$ TVD methods
  Total Variation Diminishing

$$TV(u^{n+1}) \leq TV(u^n)$$

$$TV(u^n) = \sum_i \mid u_{i+1}^n - u_i^n \mid$$



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Numerical challenges I: Range of scales

Extreme range of scales in time and space

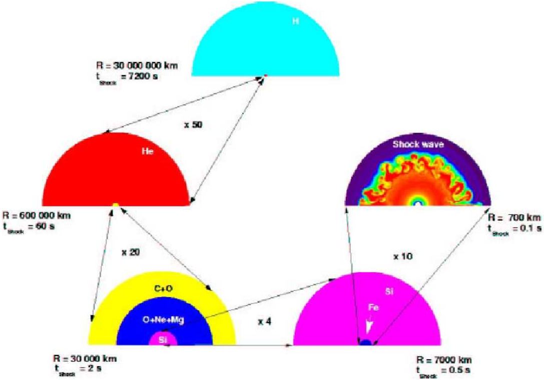
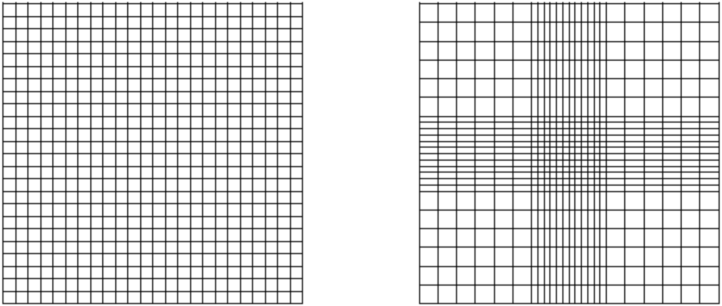


Figure: Figure taken from E. Müller

Numerical challenges II: Range of scales

Use adaptive mesh/grid



- Use finer mesh where required
- Transform irregular physical space into a regular computational space through coordinate transformation

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