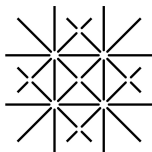


Radiative corrections to W production

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Outline

- 1 Introduction
- 2 pQFT primer: From diagram to formula
- 3 Regularisation procedures
- 4 The causal approach
- 5 Summary

What are radiative corrections?

- Leading-order diagrams are all on tree level. Corrections to them include diagrams with loops, bremsstrahlung (emission of additional final state particles), etc., which are called *radiative corrections*.

Scattering matrix element and differential cross section are related in the following way:

$$S_{tot} = S_{LO} + S_{NLO} + S_{NNLO} + \dots$$

$$\frac{d\sigma}{d\mathcal{P}} \propto |S_{tot}|^2 \propto |S_{LO}|^2 + 2 \operatorname{Re} \left(S_{LO} \cdot S_{NLO}^\dagger \right) + \dots$$

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- Relevance in W boson production: Further one's knowledge about parton distribution functions (PDFs), i.e. the probability distribution of partons (quarks and gluons) inside the hadron.

PDFs and factorisation

Factorisation of $pp \rightarrow W$ cross section in QCD-improved parton model:

$$\frac{d\sigma^{pp \rightarrow W}}{d\mathcal{P}} = \int dx_1 dx_2 \sum_{f_1, f_2} f_1(x_1, \mu^2) f_2(x_2, \mu^2) \cdot \frac{d\hat{\sigma}^{f_1 f_2 \rightarrow W}}{d\mathcal{P}}(x_1 p_1, x_2 p_2, p_W, \mu),$$

with p_i the incident proton momenta,

$f_i(x)$ the parton distribution function (PDF), i.e. the probability of finding a parton of flavour f_i and momentum fraction x ,

$d\hat{\sigma}/d\mathcal{P}$ the parton subprocess cross section, and

\mathcal{P} any appropriate set of kinematical variables;

μ defines a mass scale required for the factorisation of short- and long-range phenomena (e.g. $\mu^2 = M_W^2$).

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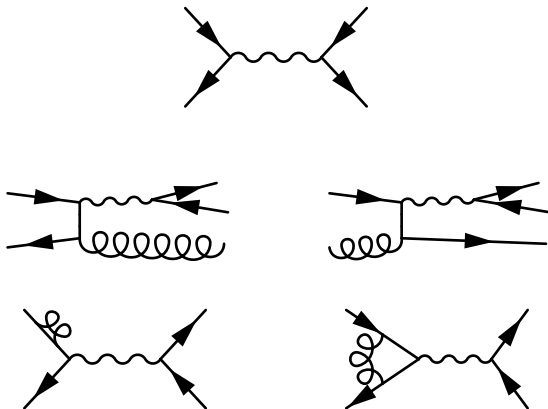
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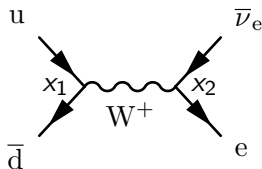
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non-perturbative (DGLAP); perturbative

LO and NLO diagrams for W production

Leading-order and possible radiative-correction diagrams to $\mathcal{O}(\alpha_s)$ contributing to d.c.s. of $q\bar{q} \rightarrow W \rightarrow \bar{\ell}\ell$:



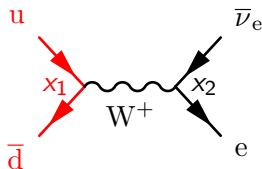
From Feynman diagram to S_{fi} 

The leading order (LO) scattering matrix element of W boson production can be written as

$$S_{fi} = \int d^4x_1 d^4x_2 \langle e \bar{\nu}_e |$$

$$\text{T} \left[: \bar{\Psi}_d(x_1) \frac{ig_w}{2\sqrt{2}} \gamma^\mu (1 - \gamma^5) V_{ud} \Psi_u(x_1) \cdot \overline{W_\mu^-(x_1)} W_\nu^+(x_2) \right.$$

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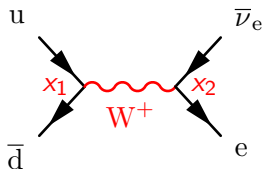
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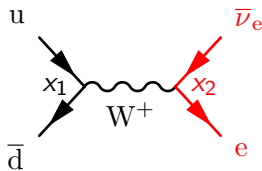
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Divergences and regularisation

Higher-order correction diagrams may contain divergences:

- UV (non-convergent integrals for $k \rightarrow \infty$),
e.g. electron self-energy $\int d^4k \frac{\gamma_\mu [\not{p} - \not{k} + im] \gamma^\mu}{(k^2 + i\epsilon)[(p-k)^2 - m^2 + i\epsilon]}$

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- IR (soft photons/gluons, integral bound $k \rightarrow 0$),
e.g. scalar vertex $\int \frac{d^4 k}{[k^2 + i\varepsilon][(k-p_1^i)^2 + i\varepsilon][(k-p_2^i)^2 + i\varepsilon]}$

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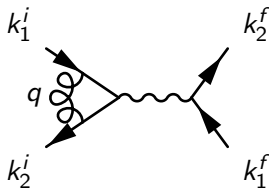
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- collinear (jet physics),
e.g. gluon emission by final state quark
 $\frac{1}{(p+k)^2 - m^2} = \frac{1}{2p \cdot k} = \frac{1}{2E_k E_p (1 - \cos \vartheta)}$

⇒ Regularisation and renormalisation procedures required.

According to the theorem by F. Bloch and A. Nordsieck, the IR divergences will cancel out if one sums up all necessary diagrams of the same order.

▶ skip W vertex

The W vertex graph



The corresponding S matrix element follows as

$$\begin{aligned}
 S_{fi} = & -g_s^2 g_w^2 \frac{V_{ud}}{12} \delta^{(4)}(k_1^i + k_2^i - k_1^f - k_2^f) \\
 & \cdot \int d^4 q \frac{1}{q^2 + i\epsilon} \bar{v}(k_2^i) \gamma_\mu \frac{\not{q} - \not{k}_2^i + m_d}{(q - k_2^i)^2 - m_d^2 + i\epsilon} \gamma^\nu (1 - \gamma^5) \\
 & \cdot \frac{\not{q} - \not{k}_1^i + m_u}{(q - k_1^i)^2 - m_u^2 + i\epsilon} \gamma^\mu u(k_1^i) \frac{1}{(k_1^f + k_2^f)^2 - M_W^2 + i\epsilon} \\
 & \cdot \bar{u}(k_2^f) \gamma_\nu (1 - \gamma^5) v(k_1^f).
 \end{aligned}$$

The W vertex graph

Simple dimensional analysis leads to

$$S_{fi} \propto \int d^4 q \frac{(q)^2}{q^6} \propto \int_0^\infty \frac{dq}{q},$$

which is logarithmically divergent, i.e. contains UV and IR divergences!

The IR divergence cancels upon summation of vertex, self-energy and bremsstrahlung diagrams; the UV divergence has to be regularised.

Regularisation procedures

The most often used methods are:

- Pauli-Villars regularisation: introduction of a fictitious heavy boson for every gauge boson whose mass goes to infinity in the final limit
- dimensional regularisation: calculating in $D = n - 2\epsilon$ dimensions instead of n , with limit $\epsilon \rightarrow 0$
- the causal approach: using causality to avoid UV divergences in the first place (a mathematically sound approach to S_{fi})

Example: Vacuum polarisation in scalar QED₃

In scalar QED₃, the Lagrangian and (perturbative) interaction Hamiltonian are given by:

$$\mathcal{L} := \partial_\mu \Phi^\dagger \partial^\mu \Phi + m^2 \Phi^\dagger \Phi + ie \Phi^\dagger \overleftrightarrow{\partial}_\mu \Phi A^\mu + e^2 \Phi^\dagger \Phi A_\mu A^\mu - \frac{1}{4} F_{\mu\nu} F^{\mu\nu}$$

$$\mathcal{H}_{int} = -ie : \Phi^\dagger \overleftrightarrow{\partial}_\mu \Phi : A^\mu \quad \text{with} \quad \Phi^\dagger \overleftrightarrow{\partial}_\mu \Phi := \Phi^\dagger \cdot \partial_\mu \Phi - \partial_\mu \Phi^\dagger \cdot \Phi$$

The Feynman diagram  leads to:

$$S_{fi} = \frac{e^2}{2} \delta^3(p - p') \left[\varepsilon_\mu(\vec{p}, \kappa) \varepsilon_\nu^*(\vec{p}', \kappa') + \varepsilon_\nu(\vec{p}, \kappa) \varepsilon_\mu^*(\vec{p}', \kappa') \right] \\ \cdot \int d^3k \frac{4k^\mu k^\nu - 2k^\mu p^\nu - 2k^\nu p^\mu + p^\mu p^\nu}{(k^2 - m^2 + i\varepsilon) \left[(k - p)^2 - m^2 + i\varepsilon \right]}$$

Example: Vacuum polarisation in scalar QED₃

The interesting part of S_{fi} is

$$\mathcal{I}^{\mu\nu}(p) := \int d^3k \frac{p^\mu p^\nu - 2k^\mu p^\nu - 2k^\nu p^\mu + 4k^\mu k^\nu}{(k^2 - m^2 + i\varepsilon) \left[(k-p)^2 - m^2 + i\varepsilon \right]},$$

which is UV divergent, according to naive dimensional analysis:

$$\mathcal{I}_3^{\mu\nu} \propto \int_0^\infty k^2 dk \frac{k^2}{k^4} = \int_0^\infty dk.$$

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The general idea behind Pauli-Villars regularisation

Introduced in 1949 by W. Pauli and F. Villars building on work by E.C.G. Stückelberg, R.P. Feynman and D. Rivier, the propagator is transformed by

$$\frac{1}{q^2 - m^2 + i\epsilon} \mapsto \frac{1}{q^2 - m^2 + i\epsilon} - \frac{1}{q^2 - \Lambda^2 + i\epsilon}$$

with $\Lambda \gg 1$ (in the end $\Lambda \rightarrow \infty$ to recover the original expression). The integrand doesn't change for small q , but is cut off smoothly for large $|q| \gtrsim \Lambda$.

Applying this procedure to

$$\mathcal{I}_3^{\mu\nu}(p) = \int d^3k k^\mu k^\nu \cdot \frac{1}{k^2 - m^2 + i\epsilon} \cdot \frac{1}{(k-p)^2 - m^2 + i\epsilon}$$

transforms it ...

Vacuum polarisation and PV

... to

$$\begin{aligned} \mathcal{I}_3^{\mu\nu}(p) &= \int d^3k \frac{(m^2 - \Lambda^2)^2 k^\mu k^\nu}{(k^2 - m^2)(k^2 - \Lambda^2)[(k-p)^2 - m^2][(k-p)^2 - \Lambda^2]} \\ &=: \mathcal{I}_3^{\mu\nu}(0) + \tilde{\mathcal{I}}_3^{\mu\nu}(p), \end{aligned}$$

with

$$\mathcal{I}_3^{\mu\nu}(0) = \int d^3k \frac{(m^2 - \Lambda^2)^2 k^\mu k^\nu}{(k^2 - m^2 + i\varepsilon)^2 (k^2 - \Lambda^2 + i\varepsilon)^2}$$

$$\begin{aligned} \tilde{\mathcal{I}}_3^{\mu\nu}(p) &= \int d^3k \frac{(m^2 - \Lambda^2)^2 k^\mu k^\nu}{(k^2 - m^2 + i\varepsilon)(k^2 - \Lambda^2 + i\varepsilon)} \\ &\cdot \left\{ \frac{-1}{(k^2 - m^2 + i\varepsilon)(k^2 - \Lambda^2 + i\varepsilon)} + \frac{1}{[(k-p)^2 - m^2 + i\varepsilon][(k-p)^2 - \Lambda^2 + i\varepsilon]} \right\}. \end{aligned}$$

Vacuum polarisation and PV

Taking the limit $\Lambda \rightarrow \infty$ of both integrals leads to

$$\begin{aligned} \mathcal{I}_3^{\mu\nu}(0) &\rightarrow 0 \\ \tilde{\mathcal{I}}_3^{\mu\nu}(p) &\rightarrow \int d^3k \frac{-(p^2 - 2kp)k^\mu k^\nu}{(k^2 - m^2 + i\varepsilon)^2 [(k-p)^2 - m^2 + i\varepsilon]}, \end{aligned}$$

the latter of which is now convergent.

Using Feynman parametrisation

$$\frac{1}{ABC} = 2 \int_0^1 d\alpha \int_0^\alpha d\beta [(\alpha - \beta)A + \beta B + (1 - \alpha)C]^{-3},$$

performing a Wick rotation ($k^0 \mapsto ik^4$) and integrating out are now straightforward.

Vacuum polarisation and PV

The final result is then

$$\mathcal{I}_3^{\mu\nu}(p) = -i\pi^2 \sqrt{p^2} \cdot \left(g^{\mu\nu} - \frac{p^\mu p^\nu}{p^2} \right) \cdot \left[\frac{2m}{\sqrt{p^2}} + \frac{p^2 - 4m^2}{2p^2} \ln \left| \frac{\sqrt{p^2} + 2m}{\sqrt{p^2} - 2m} \right| \right]$$

for the kinematic range $0 \leq p^2 \leq 4m^2$.

The general idea behind dimensional regularisation

This method, also called *dimensional continuation*, has been used for quite some time in statistical mechanics (e.g. by M.E. Fisher & D.S. Gaunt). It was first applied to quantum field theory by G. t'Hooft and others in 1972.

Changing the dimension of an integral, e.g.

$$\int \frac{d^3 q}{q^3} \mapsto \int \frac{d^D q}{q^3},$$

makes it (UV) convergent for certain values of D (above for $D < 3$).

Procedure: Calculate in $D = n - 2\epsilon$ dimensions and take the limit $\epsilon \rightarrow 0$.

If one has dimension dependent quantities (like Dirac gamma matrices) care has to be taken to properly continue them into arbitrary dimension!

Vacuum polarisation and dim. reg.

Our integral is written using Feynman parametrisation and subjected to a momentum translation ($k^\mu \mapsto k^\mu + \alpha p^\mu$):

$$\mathcal{I}_3^{\mu\nu}(p) = \int d^3k \int_0^1 d\alpha \frac{\frac{1}{3}g^{\mu\nu}k^2 + \alpha^2 p^\mu p^\nu}{[k^2 - m^2 + \alpha(1-\alpha)p^2 + i\epsilon]^2}.$$

Using the general relation

$$\int d^Dk \frac{(k^2)^r}{(k^2 - a^2 + i\epsilon)^m} = i(-1)^{r-m} \pi^{\frac{D}{2}} \frac{\Gamma(r + \frac{D}{2}) \Gamma(m - r - \frac{D}{2})}{\Gamma(\frac{D}{2}) \Gamma(m) (a^2)^{m-r-\frac{D}{2}}}$$

for the two integrals and setting $D = 3 - 2\epsilon \dots$

Vacuum polarisation and dim. reg.

... leads to

$$\int d^{3-2\epsilon}k \frac{k^2}{[k^2 - a^2 + i\epsilon]^2} = -i \frac{\pi^{\frac{3}{2}-\epsilon} \Gamma(\frac{5}{2}-\epsilon) \Gamma(-\frac{1}{2}+\epsilon)}{\Gamma(\frac{3}{2}-\epsilon) \Gamma(2) (a^2)^{-\frac{1}{2}+\epsilon}} \xrightarrow{\epsilon \rightarrow 0} 3i\pi^2 \sqrt{a^2}$$

$$\int d^{3-2\epsilon}k \frac{1}{[k^2 - a^2 + i\epsilon]^2} = i \frac{\pi^2}{\sqrt{a^2}} (\pi a^2)^{-\epsilon} \xrightarrow{\epsilon \rightarrow 0} i \frac{\pi^2}{\sqrt{a^2}}$$

and therefore

$$\mathcal{I}_3^{\mu\nu}(p) = i\pi^2 \int_0^1 d\alpha \left(g^{\mu\nu} \sqrt{a^2} + p^\mu p^\nu \frac{\alpha^2}{\sqrt{a^2}} \right),$$

with $a^2 := m^2 - \alpha(1-\alpha)p^2 \geq 0$ for $0 \leq p^2 \leq 4m^2$.

Vacuum polarisation and dim. reg.

Integrating over α gives the result

$$\mathcal{I}_3^{\mu\nu}(p) = -i\pi^2 \sqrt{p^2} \cdot \left(g^{\mu\nu} - \frac{p^\mu p^\nu}{p^2} \right) \cdot \left[\frac{2m}{\sqrt{p^2}} + \frac{p^2 - 4m^2}{2p^2} \ln \left| \frac{\sqrt{p^2} + 2m}{\sqrt{p^2} - 2m} \right| \right] + 4i\pi^2 g^{\mu\nu} m,$$

again for the kinematic range $0 \leq p^2 \leq 4m^2$.

The local term is arbitrary because of the integral's degree of divergence (see below).

Causal perturbation theory

The usual formal expression for the S-matrix is

$$S = \mathbb{1} + \sum_{n=1}^{\infty} \frac{(-i)^n}{n!} \int d^4x_1 \cdots d^4x_n \mathbb{T} [\mathcal{H}_{int}(x_1) \cdots \mathcal{H}_{int}(x_n)] .$$

Since we end up with divergences, there must be something wrong with the expression above. The problem lies with

$$\mathbb{T} [\mathcal{H}_{int}(x_1) \cdots \mathcal{H}_{int}(x_n)] \\ \stackrel{\text{def.}}{=} \sum_{\Pi} \Theta(x_{\Pi 1}^0 - x_{\Pi 2}^0) \cdots \Theta(x_{\Pi(n-1)}^0 - x_{\Pi n}^0) \cdot \mathcal{H}_{int}(x_{\Pi 1}) \cdots \mathcal{H}_{int}(x_{\Pi n}) ,$$

where one multiplies the operator-valued distributions $T_1 := -i \mathcal{H}_{int}$ with Heaviside distributions, which isn't mathematically sound (i.e. well-defined)!

Scattering matrix element and n-point distribution

E.C.G. Stückelberg, N.N. Bogoljubov, H. Epstein and V.J. Glaser noted that it's possible to avoid these divergences if one used the causal structure of the theory. Starting from a mathematically well-formed expression

$$S(g) = \mathbb{1} + \sum_{n=1}^{\infty} \frac{1}{n!} \int d^4x_1 \cdots d^4x_n T_n(x_1, \dots, x_n) \cdot g(x_1) \cdots g(x_n),$$

with T_n the n-point distribution and $g \in \mathcal{S}(\mathbb{R}^4)$, the inverse of S is given by

$$S^{-1}(g) = \mathbb{1} + \sum_{n=1}^{\infty} \frac{1}{n!} \int d^4x_1 \cdots d^4x_n \tilde{T}_n(x_1, \dots, x_n) \cdot g(x_1) \cdots g(x_n),$$

where, using $1/(1+T) = 1 + \sum_n (-T)^n$,

$$\tilde{T}_n(X) := \sum_{r=1}^n (-1)^r \sum_{P_r} T_{m_1}(X_1) \cdots T_{m_r}(X_r)$$

and the sum runs over all partitions of X into r disjoint, non-empty subsets.

The method of Epstein and Glaser

Avoiding a direct definition of T_n , one defines instead

$$A'_n(x_1, \dots, x_n) := \sum_{P_2} \tilde{T}_{n_1}(X) T_{n-n_1}(Y, x_n)$$

$$R'_n(x_1, \dots, x_n) := \sum_{P_2} T_{n-n_1}(Y, x_n) \tilde{T}_{n_1}(X),$$

where the sum runs over all partitions of $\{x_1, \dots, x_{n-1}\} = X \cup Y$, $X = \emptyset$, into two disjoint subsets with $|X| = n_1 \geq 1$ and $|Y| \leq n - 2$.

Allowing for $X = \emptyset$ yields

$$A_n(x_1, \dots, x_n) := \sum_{P_2^0} \tilde{T}_{n_1}(X) T_{n-n_1}(Y, x_n) = A'_n + T_n(x_1, \dots, x_n)$$

$$R_n(x_1, \dots, x_n) := \sum_{P_2^0} T_{n-n_1}(Y, x_n) \tilde{T}_{n_1}(X) = R'_n + T_n(x_1, \dots, x_n).$$

The method of Epstein and Glaser

Exploiting causality reveals the support properties:

$$\begin{aligned}\text{supp } R_n(Y, x_n) &\subseteq \Gamma_{n-1}^+(x_n) \\ \text{supp } A_n(Y, x_n) &\subseteq \Gamma_{n-1}^-(x_n),\end{aligned}$$

with the n -dimensional generalisation Γ_n^\pm of the closed forward (+) and backward (−) light cone.

While A_n and R_n are not known (because they contain T_n , which we're looking for), A'_n and R'_n can be constructed by induction. Their difference, however, is identical:

$$D_n = R'_n - A'_n = R_n - A_n,$$

with causal support in $\Gamma_{n-1}^+ \cup \Gamma_{n-1}^-$.

The exact way in which D_n is split into A_n and R_n using the support properties contains the crux of the whole process!

Splitting of causal distributions

Expand the distribution in external fields:

$$D_n(x_1, \dots, x_n) = \sum_k : \prod_i \bar{\Psi}(x_i) d_n^k(x_1, \dots, x_n) \prod_j \Psi(x_j) : \prod_m A(x_m),$$

with $d_n^k(x) = r(x) - a(x)$ having the same causal support as $D_n(x)$.

Analysing the behaviour of $\hat{d}_n^k(p)$ in the limit $p \rightarrow \infty$ determines its so-called *degree of divergence* ω :

- if $\omega < 0$, the trivial splitting with Heaviside distributions poses no problems and one finds

$$\hat{r}(p) = \frac{i}{2\pi} \int_{-\infty}^{\infty} dt \frac{\hat{d}(tp)}{1 - t + i\varepsilon}$$

- if $\omega \geq 0$, one has to amend this according to

$$\hat{r}(p) = \frac{i}{2\pi} \int_{-\infty}^{\infty} dt \frac{\hat{d}(tp)}{(t - i\varepsilon)^{\omega+1} (1 - t + i\varepsilon)} \quad (\star)$$

The causal approach summed up

Combining all the above, one gets the following “recipe”:

- from $T_1 = -i \mathcal{H}_{int}$ construct R'_n and A'_n inductively, from which r' and a' follow
- write down D_n and consecutively d_n^k , determine the degree of divergence ω , and split into retarded and advanced parts, r and a , respectively
- $t = r - r' = a - a'$ from which T_n follows.

Another important finding is (r and \tilde{r} are obtained using different ways of splitting):

$$r - \tilde{r} = \sum_{|a| \leq \omega} C_a D^a \delta(x),$$

with $C_a \in \mathbb{R}$ and D^a the multi-index differentiation operator.

In the case of $\omega \geq 0$, r is determined only up to local terms, which have to be fixed using physical constraints.

The causal approach to vacuum polarisation

Starting all over again, we remember the interaction Hamiltonian

$$T_1(x) := -i \mathcal{H}_{int}(x) = -e : \Phi^\dagger(x) \overleftrightarrow{\partial}_\mu \Phi(x) : A^\mu(x) \text{ and write thus}$$

$$A'_2(x, y) = -e^2 : \Phi^\dagger(x) \overleftrightarrow{\partial}_\mu \Phi(x) : : \Phi^\dagger(y) \overleftrightarrow{\partial}_\nu \Phi(y) : : A^\mu(x) A^\nu(y) :,$$

$$R'_2(x, y) = -e^2 : \Phi^\dagger(y) \overleftrightarrow{\partial}_\mu \Phi(y) : : \Phi^\dagger(x) \overleftrightarrow{\partial}_\nu \Phi(x) : : A^\mu(x) A^\nu(y) :,$$

and

$$D_2(x, y) = R'_2(x, y) - A'_2(x, y).$$

Using Wick's theorem on normal ordering, this leads to, e.g., the first term in A'_2

$$\begin{aligned} & : \Phi^\dagger(x) \partial^\mu \Phi(x) : : \Phi^\dagger(y) \partial^\nu \Phi(y) : := \Phi^\dagger(x) \partial^\mu \Phi(x) \Phi^\dagger(y) \partial^\nu \Phi(y) : \\ & + : \Phi^\dagger(x) \partial^\nu \Phi(y) : i \partial_x^\mu D^{(+)}(x-y) - : \partial^\mu \Phi(x) \Phi^\dagger(y) : i \partial_y^\nu D^{(-)}(y-x) \\ & \qquad \qquad \qquad + \partial_x^\mu D^{(+)}(x-y) \partial_y^\nu D^{(-)}(y-x), \end{aligned}$$

with $D^{(\pm)}$ the positive/negative frequency part of the Pauli-Jordan distribution.

The causal approach to vacuum polarisation

Since normal ordering of the terms in D_n doesn't obscure causal support, one can do distribution splitting for every diagram separately.

We're only interested in the vacuum polarisation, whose D_2 is

$$\begin{aligned}
 D_2(x, y) = & -e^2 : A_\mu(x) A_\nu(y) : \\
 & \cdot \left[\partial_y^\nu D^{(+)}(y-x) \partial_x^\mu D^{(-)}(x-y) - \partial_y^\nu D^{(-)}(y-x) \partial_x^\mu D^{(+)}(x-y) \right. \\
 & + \partial_y^\mu D^{(+)}(y-x) \partial_y^\nu D^{(-)}(x-y) - \partial_x^\mu D^{(-)}(y-x) \partial_y^\nu D^{(+)}(x-y) \\
 & - \partial_x^\mu \partial_y^\nu D^{(+)}(y-x) D^{(-)}(x-y) + \partial_x^\mu \partial_y^\nu D^{(-)}(y-x) D^{(+)}(x-y) \\
 & \left. - D^{(+)}(y-x) \partial_x^\mu \partial_y^\nu D^{(-)}(x-y) + D^{(-)}(y-x) \partial_x^\mu \partial_y^\nu D^{(+)}(x-y) \right].
 \end{aligned}$$

For further calculation, we change to momentum space and write:

$$D_2(x, y) =: d^{\mu\nu}(x, y) : A_\mu(x) A_\nu(y) : \quad \text{with} \quad \hat{d}^{\mu\nu}(k) =: \frac{e^2}{(4\pi)^2} \left(g^{\mu\nu} - \frac{k^\mu k^\nu}{k^2} \right) \hat{d}(k)$$

The causal approach to vacuum polarisation

After some calculation, one finds

$$\hat{d}(k) = \frac{\sqrt{k^2}}{2} \Theta(k^2 - 4m^2) \operatorname{sgn}(k_0) \left(1 - \frac{4m^2}{k^2}\right),$$

who behaves like $|k|$ for $k \rightarrow \infty$. Therefore, $\omega = 1$.

Remembering eq. (★), we now easily write

$$\begin{aligned} \hat{r}(k_0) &= \frac{ik_0^2}{2\pi} \int_{-\infty}^{\infty} dp_0 \frac{|p_0| \Theta(p_0^2 - 4m^2) \operatorname{sgn}(p_0)}{(p_0 - i\varepsilon)^2 (k_0 - p_0 + i\varepsilon)} \cdot \frac{1}{2} \left(1 - \frac{4m^2}{p_0^2}\right) \\ &= \frac{ik_0^2}{4\pi} \int_{4m^2}^{\infty} \frac{ds}{s^{3/2}} \frac{s - 4m^2}{k_0^2 - s + i\varepsilon k_0}. \end{aligned}$$

The causal approach to vacuum polarisation

From R'_2 follows (with similar definitions as for D_2)

$$\hat{r}'(k_0) = -\frac{k_0}{2} \Theta(k_0^2 - 4m^2) \Theta(-k_0) \left(1 - \frac{4m^2}{k_0^2} \right)$$

and therefore

$$\hat{t}(k_0) = \hat{r}(k_0) - \hat{r}'(k_0) = \frac{ik_0^2}{4\pi} \int_{4m^2}^{\infty} \frac{ds}{s^{3/2}} \frac{s - 4m^2}{k_0^2 - s + i\varepsilon}.$$

Integrating out and using the definition of $\hat{t}^{\mu\nu}$ (similar to $\hat{d}^{\mu\nu}$) gives:

$$\hat{t}^{\mu\nu}(k) = -2 \frac{ie^2}{(4\pi)^3} \left(g^{\mu\nu} - \frac{k^\mu k^\nu}{k^2} \right) \sqrt{k^2} \left[\frac{2m}{\sqrt{k^2}} + \frac{k^2 - 4m^2}{2k^2} \ln \left| \frac{\sqrt{k^2} + 2m}{\sqrt{k^2} - 2m} \right| \right].$$

Summary

- radiative corrections are higher-order corrections to tree-level graphs
- they may contain UV, IR, and collinear singularities \Rightarrow need for regularisation and renormalisation theory
- possible regularisation procedures:
 - Pauli-Villars
 - dimensional regularisation
 - the causal approach
- causal approach the most sensible from a mathematical point of view