BLACK HOLES AND QUANTUM MECHANICS*

Gerard ’t Hooft

Institute for Theoretical Physics
Utrecht University
and
Spinoza Institute
Postbox 80.195
3508 TD Utrecht, the Netherlands
e-mail: g.thooft@uu.nl
internet:  http://www.phys.uu.nl/~thooft/

Abstract

After a brief review of quantum black hole physics, it is shown how the dynamical properties of a quantum black hole may be deduced to a large extent from Standard Model Physics, extended to scales near the Planck length, and combined with results from perturbative quantum gravity. Together, these interactions generate a Hilbert space of states on the black hole horizon, which can be investigated, displaying interesting systematics by themselves.

To make such approaches more powerful, a study is made of the black hole complementarity principle, from which one may deduce the existence of a hidden form of local conformal invariance.

Finally, the question is raised whether the principles underlying Quantum Mechanics are to be sharpened in this domain of physics as well. There are intriguing possibilities.

February 20, 2010

*Schladming, Austria, March 2010
1. Introduction: Schwarzschild metric and Kruskal-Szekeres coordinates

The Standard Model of Elementary Particles emerged empirically as an extremely efficient way to describe all particles and forces that have been detected experimentally thus far. However, in these explorations, the gravitational force was far too weak to be taken into account. In our attempts to include gravity in theory, we encounter the difficulty that (real or virtual) black holes might form. Their very nature causes unforeseen difficulties when we try to formulate a theory consistent with quantum mechanics. Various approaches will be considered in these notes, although we leave string theories for others to cover. In the work to be presented here, a pivotal theme is the notion that a book keeping principle must be devised to handle discrete bits of information that describe the dynamical processes at the Planck scale.

We introduce the metric of space-time in the vicinity of a black hole with mass $M$, which is defined by specifying the locally Lorentz covariant distance-squared $d\!s^2$ between two points, $x$ and $x+dx$, the separation $dx$ being infinitesimal:

$$d\!s^2 = -\left(1 - \frac{2M}{r}\right)dt^2 + \frac{dr^2}{1-2M/r} + r^2 d\Omega^2,$$

(1.1)

where spherical coordinates $t, r, \theta, \varphi$ were chosen, $d\Omega^2 \overset{\text{def}}{=} d\theta^2 + \sin^2 \theta \; d\varphi^2$, and the radial coordinate $r$ was chosen such that the angular component in this expression coincides with that of empty space-time. Furthermore, we absorbed Newton’s constant $G_N$ into the definition of the mass: $M \overset{\text{def}}{=} G_N m_{BH}$, or rather, units were chosen such that $G_N = 1$.

![Figure 1](image_url)

Figure 1: a. Kruskal coordinates $x$ and $y$ for the pure black hole. Dotted lines are $r = C^{\text{inst}}$ lines. Blue region: the forbidden $r < 0$ region. b. Spacetime for a black hole formed by matter moving in, as described by Eqs. (1.4)–(1.9). Ingoing matter is indicated by green region.

The Kruskal-Szekeres coordinates $(x, y, \theta, \varphi)$ are defined by the equations

$$\left(\frac{r}{2M} - 1\right)e^{r/2M} = xy;$$
In terms of these coordinates, one calculates
\[
\frac{dx}{x} + \frac{dy}{y} = \frac{dr}{2M(1 - 2M/r)};
\]
\[
\frac{dx}{x} - \frac{dy}{y} = \frac{dt}{2M};
\]
\[
ds^2 = \frac{32M^3}{r} e^{-v/2M} dx dy + r^2 d\Omega^2,
\]
and one notices that the singularity at \( r \to 2M \), or \( xy = 0 \), has disappeared. Only the singularity at \( r = 0 \), or \( xy = -1 \) remains.

To get a rough idea how black holes can be formed by matter, we consider the case when massless non-interacting particles move inwards with the speed of light, in a spherically symmetric configuration (later one may decide to relax on these conditions by allowing for all sorts of perturbations, to observe that the net result, the formation of a black hole, is often not affected). The metric is then
\[
ds^2 = 2A(x,y) dx dy + r^2(x,y) d\Omega^2,
\]
where the functions \( A \) and \( r \) are yet to be determined. If we assume that the stress-energy-momentum tensor \( T_{\mu\nu} \) has only one non-vanishing component \( T_{xx} \), with \( \frac{\partial}{\partial y}(r^2 T_{xx}) = 0 \), then we are describing massless matter moving inward.

Using a short hand notation, \( \partial r/\partial x = r_x \), etc., we define a new function \( M(x,y) \) as
\[
A = \frac{2r r_x r_y}{r - 2M},
\]
in terms of which the Einstein equations become surprisingly simple. Since we demand that \( T_{xy} = T_{\theta\theta} = 0 \), the Ricci curvature components \( R_{xx} \) and \( R_{\theta\theta} \) also have to vanish. This gives
\[
M_{xy} = -\frac{2M_x M_y}{r - 2M}.
\]
These can both be integrated to give
\[
2M_x r_x = g(x) \left( 1 - \frac{2M}{r} \right), \quad 2M_y r_y = h(y) \left( 1 - \frac{2M}{r} \right),
\]
where \( g(x) \) and \( h(y) \) are arbitrary functions. Since
\[
8\pi G_N T_{xx} = \frac{2g(x)}{r^2}, \quad \text{and} \quad 8\pi G_N T_{yy} = \frac{2h(y)}{r^2},
\]
we now demand that \( g(x) \geq 0 \) and \( h(y) = 0 \). Indeed, the function \( h(y) \) would describe particles moving outwards. This implies that \( M(x,y) = M(x) \) is an increasing function of \( x \).
Let us assume that the particles begin falling in when \( x \geq 0 \), then at \( x < 0 \) and at spacelike infinity, we can write

\[
\begin{align*}
    r &= \frac{1}{\sqrt{2}} (x + y); \\
    t &= \frac{1}{\sqrt{2}} (x - y), \\
    ds^2 &\rightarrow dr^2 - dt^2 + r^2 d\Omega^2. \\
\end{align*}
\]

whereas the metric after the collision event, \( x > x_1 \), coincides with (1.3). It can also easily be shown that that parameter \( M \) indeed corresponds to the total mass of the inwards moving particle, as established by an observer at \( t \rightarrow -\infty \).

Close to the horizon, \( r \approx 2M \), one may use coordinates that are approximately locally flat:

\[
\begin{align*}
    \sqrt{A}x^+ &= \frac{1}{\sqrt{2}} (Z + T) = \frac{\theta}{\sqrt{2}} e^{\tau/2}, \\
    x^- &= \sqrt{A}y = \frac{1}{\sqrt{2}} (Z - T) = \frac{\theta}{\sqrt{2}} e^{-\tau/2}, \\
    \tilde{x} &= \binom{X}{Y} = \binom{r(\theta - \frac{1}{2}\pi)}{r\varphi}. \\
\end{align*}
\]

Here,

\[
    ds^2 \rightarrow 2dx^+ dx^- + d\tilde{x}^2 = dg^2 = d\theta^2 - \varphi^2 d\tau^2 + d\tilde{x}^2. 
\]

The coordinates \( (\tau, \varphi, \tilde{x}) \) are called Rindler coordinates. Ignoring the local curvature at the horizon simplifies our considerations.

### 2. Hawking radiation

For what follows it is important to understand what causes the Hawking effect. For more complete derivations, we refer to the literature\[1\],[2]. Here, we give a brief summary. What is usually assumed (for an exception, see ref.\[3\]), is that the expectation values for fields, such as a scalar field \( \phi(x) \), are the same for all observers. In practice, these fields are space-time dependent operators that can be applied to states in Hilbert space. Let there now be a time coordinate \( t \), used by one particular observer, and let it be such that, in some reasonable approximation, the metric and other physical background data are invariant under time translations:

\[
    H(t) = H(t + a), \tag{2.1}
\]

where \( H \) is the Hamiltonian that describes the time evolution:

\[
    \frac{d}{dt}|j\rangle = -iH|\psi\rangle. \tag{2.2}
\]

Let \( a(\vec{x}, \omega) \) be the Fourier transforms of the fields \( \phi(\vec{x}, t) \) with respect to time:

\[
    \phi(\vec{x}, t) = (2\pi)^{-\frac{1}{2}} \int_{-\infty}^{\infty} a(\vec{x}, \omega)e^{-i\omega t}, \tag{2.3}
\]
where we did not bother to Fourier transform also the space coordinates $\vec{x}$. One then has, quite generally,

$$\dot{\phi}(t) = -i[\phi, H], \quad [a(\omega), H] = -i\omega a(\omega), \quad H(a(\omega)|\psi\rangle) = a(\omega)(H - \omega)|\psi\rangle.$$  \hspace{1cm} (2.4)

Thus, if $\omega$ is positive, the operator $a(\omega)$ always lowers the energy of a state by an amount $\omega$; it is an annihilation operator that removes a particle with energy $\omega$ from the state. Conversely, the negative $\omega$ operators, $a(-\omega) = (a(\omega))^\dagger$, are creation operators.

The vacuum state $|\text{vac}\rangle$ is the state out of which no particle can be removed:

$$\phi(\omega)|\text{vac}\rangle = 0, \quad \text{iff} \quad \omega > 0.$$  \hspace{1cm} (2.5)

Now an observer who is freely falling into a black hole, does not experience divergent forces while going through the horizon. Classically, at least, the situation at that point appears to be completely regular, locally. This is why, also in quantum mechanics, one has no reason to expect divergent physical effects when this observer passes the horizon. Consider now the fields $\phi(x)$ inside a small box passing the horizon, just where the observer is, so that the Rindler approximation, Eq. (1.10), is justified. Fourier transforming such a field, we expect that the observer experiences a state $|\text{vac}_1\rangle$ where no particles can be annihilated. In particular, at very high values of $\omega$, no particles can be annihilated: $a(\omega)|\text{vac}_1\rangle = 0$. This completely characterizes the state of the quantum particles as observed by this observer. Thus, one can compute all correlation functions $\langle \text{vac}|\phi(x_1)\phi(x_2)\cdots\phi(x_n)|\text{vac}_1\rangle$, and they should now be the same for all other observers (of course, tensor fields will have to be transformed as tensors go).

Now one can consider a “Rindler observer”. This observer uses the Schwarzschild $t$ coordinate as his time coordinate. This corresponds to the coordinate $\tau$ in Eqs.(1.10) and (1.11), while the freely falling observer uses the time coordinate $T$. For the Rindler observer, spacetime near the horizon splits into two regions: region $I$ has $\varrho > 0$, and region $II$ has $\varrho < 0$. These two regions appear to be separated by a wall that cannot be passed because the gravitational field appears to be infinite there. A somewhat lengthy but straightforward calculation now gives the following result. In region $I$, we have

$$\phi_I(\varrho, \tau) = \int_0^\infty d\omega e^{-i\omega\tau} \int \frac{d^2k e^{i\vec{k}\vec{x}}}{\sqrt{2(2\pi)^4}} K(\omega, \frac{1}{2}\mu\varrho) \left( a_2(\tilde{k}, \omega) + e^{-\pi\omega} a_2^\dagger(-\tilde{k}, -\omega) \right) + \text{h.c.},$$  \hspace{1cm} (2.6)

while in region $II$, where $\varrho < 0$,

$$\phi_I(\varrho, \tau) = \int_0^\infty d\omega e^{-i\omega\tau} \int \frac{d^2k e^{i\vec{k}\vec{x}}}{\sqrt{2(2\pi)^4}} K(\omega, \frac{1}{2}\mu\varrho) \left( a_2(\tilde{k}, \omega) + e^{\pi\omega} a_2^\dagger(-\tilde{k}, -\omega) \right) + \text{h.c.},$$  \hspace{1cm} (2.7)

writing $\mu = \sqrt{k^2 + m^2}$ and

$$K(\omega, \alpha) = \int_0^\infty \frac{ds}{s} s^{i\omega} e^{i\alpha(-s+1/s)}.$$  \hspace{1cm} (2.8)
The operator $a_2(\tilde{k},\omega)$ is a linear combination of the annihilation operators as would be used by the freely falling observer\cite{2}:

$$a_2(\tilde{k},\omega) = (2\pi)^{-1/2} \int_0^\infty \frac{dk^3}{\sqrt{k^0}} a(\tilde{k},k^3) e^{i\omega \ln \left( \frac{k^3 + k^0}{k^0} \right)} , \quad k^0 = \sqrt{k^3 + \mu^2} , \quad (2.9)$$

where $a(\tilde{k},k^3)$ is the space Fourier transform of the operator $a(\tilde{x},\omega)$ introduced in Eq. (2.3), with $w > 0$. Note that, if $\alpha > 0$, the kernel function $K$ obeys

$$K(\omega,-\alpha) = e^{-\pi \omega} K(\omega,\alpha) . \quad (2.10)$$

One now has to perform the Bogolyubov transformation, when $\omega > 0$,

$$\begin{pmatrix} a_I(\tilde{k},\omega) \\ a_{II}(\tilde{k},\omega) \\ a_I^\dagger(-\tilde{k},\omega) \\ a_{II}^\dagger(-\tilde{k},\omega) \end{pmatrix} = \frac{1}{\sqrt{1 - e^{-2\pi \omega}}} \begin{pmatrix} 1 & 0 & 0 & e^{-\pi \omega} \\ 0 & 1 & e^{-\pi \omega} & 0 \\ e^{-\pi \omega} & 0 & 0 & 1 \\ e^{-\pi \omega} & 1 & 0 & e^{-\pi \omega} \end{pmatrix} \begin{pmatrix} a_2(\tilde{k},\omega) \\ a_2(\tilde{k},-\omega) \\ a_2^\dagger(-\tilde{k},\omega) \\ a_2^\dagger(-\tilde{k},-\omega) \end{pmatrix} , \quad (2.11)$$

to see that, in region $I$, the fields $\phi$ only depend on $a_I$ and $a_I^\dagger$, while in region $II$, the fields only depend on $a_{II}$ and $a_{II}^\dagger$. All sets of our creation and annihilation operators obey similar commutation rules:

$$[a(\tilde{k}),a_I^\dagger(\tilde{k}')] = \delta^3(\tilde{k} - \tilde{k}') , \quad [a_2(\tilde{k},\omega),a_2^\dagger(\tilde{k}',\omega')] = \delta^2(\tilde{k} - \tilde{k}')\delta(\omega - \omega') , \quad [a_I(\tilde{k},\omega),a_{II}^\dagger(\tilde{k}',\omega')] = \delta(\omega - \omega')\delta^2(\tilde{k} - \tilde{k}') ; \quad [a_I,a_{II}] = [a_I,a_I^\dagger] = 0 . \quad (2.12)$$

From these rules, we derive that all observables in region $I$ only depend on $a_I$ and $a_I^\dagger$, while they commute with all observables in region $II$, which only depend on $a_{II}$ and $a_{II}^\dagger$.

The most important feature of this calculation is the emergence of the factors $e^{\pm \pi \omega}$, which can be related to the effect of analytic continuations around the origin if Rindler space-time, or the origin of the Kruskal-Szekeres coordinate frame, where the black hole horizon is. The Lorentz boost operator for a freely falling observer acts as a Hamiltonian for the Rindler observer:

$$M_L = \int_{-\infty}^{\infty} d\omega a_I^\dagger(\tilde{k},\omega)a_2(\tilde{k},\omega)$$

$$= \int_0^\infty d\omega \left( a_I^\dagger(\tilde{k},\omega)a_I(\tilde{k},\omega) - a_{II}^\dagger(\tilde{k},\omega)a_{II}(\tilde{k},\omega) \right)$$

$$= H_{\tilde{K}} - H_{II} . \quad (2.13)$$

In a sense, it appears that time “runs backwards” in region $II$.

The vacuum state $|\text{vac}_1\rangle$ for the freely falling observer is defined by

$$a_2(\tilde{k},\omega)|\text{vac}_1\rangle = 0 , \quad (2.14)$$
at all values (positive or negative) of $\omega$. This implies that
\begin{align*}
a_I(\tilde{k}, \omega)|\text{vac}_1\rangle &= e^{-\pi \omega} a_I^\dagger(-\tilde{k}, \omega)|\text{vac}_1\rangle, \\
a_{II}(\tilde{k}, \omega)|\text{vac}_1\rangle &= e^{-\pi \omega} a_I^\dagger(-\tilde{k}, \omega)|\text{vac}_1\rangle,  \quad (2.15)
\end{align*}
and these can be solved, finding that in each momentum mode, the state takes the form
\begin{equation}
|\text{vac}_1\rangle = \sqrt{1 - e^{2\pi \omega}} \sum_{n=0}^{\infty} |n\rangle_I|n\rangle_{II} e^{\pi n \omega},  \quad (2.16)
\end{equation}
where $|n\rangle$ is the $n$ particles state, having energy $n\omega$ in natural units. As, in wedge $I$, the state in wedge $II$ cannot be observed, one finds that the probability of observing $n$ particles is
\begin{equation}
e^{\beta E_n}, \quad E_n = n\omega, \quad \beta = 1/kT = 2\pi, \quad (2.17)
\end{equation}
where $T$ is the Hawking temperature. Re-inserting the time units for the Schwarzschild metric, one arrives at
\begin{equation}
T = \frac{1}{8\pi M}, \quad (2.18)
\end{equation}
where $M$ is the black hole mass.

From this result, a very important conclusion is drawn: one can define the entropy $S$ of a black hole by the equation
\begin{equation}
TdS = dQ = dM; \quad dS = 8\pi MdM; \quad S = 4\pi M^2 + C_{\text{нст}}, \quad (2.19)
\end{equation}
The constant is usually assumed to be small, but it is unknown. According to statistical physics, the entropy is related to the density of the quantum states. Apparently, a black hole must possess a spectrum of quantum states. The density of this spectrum must be $W = e^S$, which is given by Eq. (2.19), apart from an overall multiplicative constant.

3. Horizon dynamics

As black holes should play an important role in the dynamics of Nature at Planck scales, we should be able to identify these quantum states, and determine the overall constant. We would like to settle these questions by a more detailed analysis of the equivalence axiom in General Relativity. It should be possible to use that to unravel the quantum aspects of a horizon.

We must assume that the thermal nature of the black hole, as found in the previous section, is only an approximation referring to its equilibrium state. It must be possible to consider single, pure quantum states of the black hole as well, without reference to the unphysical domain $II$, where no observations can be performed. We notice that, classically, this region can be transformed away completely. A transformation
\begin{equation}
t \rightarrow t + f(r), \quad f(r) = \log(r - 2M) \quad \text{if} \quad r > 2M; \quad f(r) = \infty \quad \text{if} \quad r \leq 2M, \quad (6.1)
\end{equation}
removes region II entirely.

Therefore, we should concentrate on physics in region I, and use the physical principle of equivalence of all general coordinate frames. A promising strategy is the following. We assume a Hilbert space of \( n \approx e^S \) states, where the exact state a black hole is in is determined by all details of its past history. If the evolution is sufficiently chaotic, then the thermal state should be good enough to describe the average situation, but not the details of the dynamics. Let there be a description of one of the initial states, \( | \text{in} \rangle \), and a description of the corresponding final state, \( | \text{out} \rangle \). At time \( t \approx 0 \), we now allow one single, extra particle to enter the black hole. In the local Rindler frame at the horizon, we specify its ingoing 4-momentum, which, for future convenience, we call \( \delta p^\text{in} \). Then, we compare two different in-states: \( | \text{in} \rangle \) and \( | \text{in}' \rangle = | \text{in} + \delta p^\text{in} \rangle \). How will the corresponding states \( | \text{out} \rangle \) and \( | \text{out}' \rangle \) be related? We take the ingoing particle to enter at the transverse position \( \tilde{x} = 0 \).

The main effect of the particle \( \delta p \) on the out state will be gravitational, and can be calculated. At Schwarzschild time \( t \gg 0 \), the ingoing particle \( \delta p \) will be strongly Lorentz boosted, as seen by a later observer falling in. The momentum in the ingoing direction, \( \delta p^\text{in} \), will be strongly enhanced. For the freely falling observer in Rindler space, the local space-time metric will be modified into

\[
d ds^2 = d\tilde{x}^2 + 2dx^+(dx^- + \theta(x^+)\delta p^- df(\tilde{x})) ,
\]

where the displacement function \( f(\tilde{x}) \) obeys the equation

\[
\ddot{f}(\tilde{x}) = -8\pi G_N \delta^2(\tilde{x}) , \quad f(\tilde{x}) = -4G_N \log |\tilde{x}| .
\]

What this tells us is that, due to a gravitational dragging effect from the extra ingoing particle, all outgoing particles will be dragged inwards along the \( x^- \) axis, by an amount \( f(\tilde{x}) \delta p^- \). Though at time \( t \approx 0 \) this dragging effect may seem to be tremendously tiny, it is enhanced by an exponentially diverging Lorentz boost factor, as time proceeds, just like \( \delta p^- \) itself, as regarded by the later observers.

The shift of all particles in the \( | \text{out} \rangle \) state, can be conveniently expressed in terms of an operator in its Hilbert space:

\[
| \text{out}' \rangle = e^{-i\int d^2\tilde{x}\{\delta x^- P^+_{\text{out}}(\tilde{x})\}}| \text{out} \rangle , \quad \delta x^- = \delta p^- f(\tilde{x}) ,
\]

where \( P^+_{\text{out}}(\tilde{x}) \) is the quantum generator of a shift localized at \( \tilde{x} \), which also measures the total outgoing momentum at the point \( \tilde{x} \). This was when one extra particle enters with momentum \( \delta p^- \) at the point \( \tilde{x}' = 0 \). We can subsequently let more particles enter, at different points \( \tilde{x}' \). Taking all these expressions together, we find the operator expression

\[
| \text{out}' \rangle = e^{-i\int d^2\tilde{x}d^2\tilde{x}'\{P^-_{\text{in}}(\tilde{x}')f(\tilde{x} - \tilde{x}')P^+_{\text{out}}(\tilde{x})\}}| \text{out} \rangle ,
\]

where \( P^-_{\text{in}} \) is the total momentum operator of all ingoing particles. Notice the time reversal invariance that this expression shows.
What is the Hilbert space in which these operators act? We can interpret the operator in Eq. (3.5) as an $S$-matrix connecting the in- to the out states. The operator is unitary, provided we characterize the in- and out states exclusively by specifying the radial momentum distribution. This is unlike the standard procedure in the quantization of fields, where not the total momenta, but the particles themselves define the states. It is, however, quite similar to the procedure in string theories, where in- and out going string states are represented by vertex insertions $e^{ipx}$, where $p$ is distributed over the string world sheet. The $n$ particle state is then represented by assuming the inserted momentum distribution to consist of $n$ delta peaks across the string world sheet. Their positions on the world sheet are then subsequently integrated over, to obtain the Koba-Nielsen amplitude. In our case, we can represent the $n$ particle state the same way.

This approach to the analysis of horizon dynamics can be refined a lot. A natural next step is to consider the way in which electric charges of the ingoing particles affect the outgoing ones. The electromagnetic fields due to a charged particle moving in the negative $z$ direction with great velocities along the transverse position $\tilde{x} = 0$, can be readily calculated. They turn out to vanish both in front of the particle and behind it, but in a transverse plane moving alongside the particle with the velocity of light (a plane Cerenkov shock wave), the vector fields $A_+(x^+)$ are delta peaked. Writing $A_+(x) = \partial_+ \Lambda(x)$, where
\begin{equation}
\Lambda(x) = \theta(x^+) \frac{Q}{2\pi} \log |\tilde{x}|,
\end{equation}
gives this field configuration. It is not a pure gauge, because we impose all other components of $A_\mu(x)$ to vanish. Again, we write this as a Green function,
\begin{equation}
\Lambda = \theta(x^+) \Lambda(\tilde{x}) ; \quad \Lambda(\tilde{x}) = -f_1(\tilde{x})Q , \quad \tilde{\partial}^2 f_1(\tilde{x}) = -\delta^2(\tilde{x}) .
\end{equation}

Consider now a charge distribution $\varrho_{in}(\tilde{x})$ for the ingoing particles. This causes the wave functions for all outgoing particles to be gauge rotated with the gauge generator $\Lambda(\tilde{x})$. The total gauge rotation is
\begin{equation}
e^{i \int d^2 \tilde{x} \Lambda(\tilde{x})} \varrho_{out}(\tilde{x}) = e^{-i \int d^2 \tilde{x} \int d^2 \tilde{x}' f_1(\tilde{x} - \tilde{x}') \varrho_{in}(\tilde{x}') \varrho_{out}(\tilde{x})} .
\end{equation}

This should be multiplied with the gravitational operator (3.5). Remarkably, here, the charge distribution behaves very much like a Kaluza Klein momentum in the 5th dimension.

The following program should now be considered. It still has not yet been completed, and my hope is that someone will pick it up. Repeating the procedure for other types of fields and interactions, it is possible to represent other types of interactions between the in- and the out particles. The first candidates are[4]:

* generalize the Maxwell case to include non-Abelian gauge fields;
* handle the interactions due to scalar fields, including the Higgs effect;
* add Dirac fields;
* now reconsider the gravitational contribution, this time also including the effects of the transverse gravitational forces;

* check the effects of scaling, using the renormalization group. A consistent picture should emerge;

* address the question of the micro states. Up till here, this approach did not allow us to come close to the Planck scale, which is why the Hilbert space of states considered up till now appears to be continuous, but it is obvious that the states should be discrete, in accordance to the Hawking entropy law.

We found that, while doing this, one could first also consider the effects of magnetic monopole charges. Including them is as straightforward as our treatment of the electric charges. The non-Abelian case, on the other hand, is a bit more tricky because one cannot ignore the higher order quantum corrections, and the sources, in the form of non-Abelian charges, are obviously not gauge invariant. One could use the approach of the maximally Abelian gauge condition, which turns non-Abelian Yang-Mills fields into a set of Abelian fields, augmented with the presence of Abelian magnetic monopoles.

The effects of scalar fields can be understood. At first sight, it may seem that scalar fields leave no traces on the horizon, due to the no-hair theorem. This, however, is not true. If, for instance, the Higgs mechanism is active, the gauge field forces become short-range, and this certainly has an effect on expressions such as Eq. (3.8), where the Green function $f(\tilde{x})$ will be replaced by one with exponential damping. It turns out that the following happens. Scalar fields, being Lorentz invariant, keep their values on the horizon unchanged as Schwarzschild time goes by. Their presence does affect other fields, for instance by the Higgs effect. The effective scalar field action, $\Gamma(\phi)$, will determine the probabilities that certain scalar field configurations occur. They then produce the physical interaction terms of the action of other fields. What this means is that, besides the interacting parts of the black hole Hilbert space, described by the dynamical evolution of states $|\text{in}\rangle$ into states $|\text{out}\rangle$, see Eqs. (3.5) and (3.8), we also have static parts. The states are degenerated, and the distribution of these static components is determined by the scalar fields and their effective action.

All fields that are not Lorentz invariant, such as also the fermionic fields, contribute to the evolution of the black hole. Since, eventually, all black holes should evolve, the scalar fields will not be absolutely scalar; at very small distance scales, they probably dissolve in the form of bound states of less trivial fields.

If there should be any doubt how to proceed in this program, one should realize that we are talking about the scattering matrix of in-fields into out fields at extremely high c.m. energies. For all quantum field systems that are asymptotically free, or at least renormalizable, the extremely high energy limit should be accessible, so that models such as the Standard Model, are suitable for this treatment. Our programme thus asks for using standard quantum field theory, augmented with perturbative quantum gravity, to disentangle the dynamical properties of the black hole microstates on the horizon.
4. Black Hole Complementarity

On large time scales, Hawking radiation should have an effect on the metric. This has caused considerable amounts of confusion. According to an ingoing observer, there is no Hawking radiation, and the metric is smooth across the horizon. Yet an outside observer may see that a black hole gradually shrinks due to mass/energy loss. How do we reconcile these apparently conflicting observations?

*Black hole complementarity*\(^{[5]}[6][7][8]\) refers to the fact that observers who stay outside the black hole can see the Hawking particles, including the effect they have on the space-time metric, while observers entering the black hole can neither see these particles, nor the effect they have on the metric. On the other hand, these observers do have to add all particles that entered, and will enter, the black hole to obtain a meaningful description of what is going on there. These differences are caused by the fundamental difference in the way the two sets of observers experience the notion of time.

It was proposed to bring a slight twist in this notion of complementarity\(^{[9]}\). Rather than distinguishing observers who go into a black hole from the ones staying out, we note that the ingoing observer sees everything that went into the black hole, whereas the second observer sees all objects going out – the Hawking radiation. Thus reformulated, complementarity really refers to the mapping from the in-states to the out-states. This way, we make contact with the previous section; the black hole scattering matrix, which we began constructing in Eq. (3.5), connects inside to outside observers.

It is instructive then, to consider the special case that the black hole was formed by the collapse of a single shell of massless non-interacting particles moving in radially, because this case has a simple exact analytical solution: outside the shell we have the Schwarzschild metric, inside the shell space-time is flat, and the two regions are glued together in such a way that the metric is continuous. Thus, \(M(x) = M_0 \theta(x)\) in Eqs. (1.5)) and \(g(x) = 2M_0 r_0 \delta(x)\) in Eq. (1.7). Furthermore, we take the case that the outgoing Hawking particles also form a single shell of matter. Since, actually, the Hawking particles have a thermal spectrum, it is highly unlikely that these particles emerge as a single shell, but it is not impossible. The shell is suppressed by large Boltzmann factors, but this just means that it occupies a very tiny portion of phase space; that is not relevant to our procedure. Later, one may replace the in- and outgoing matter configurations by more realistic ones. Thus, for simplicity, the in-configuration is described by Fig. 2\(a\), and the out-configuration by 2\(b\).

According to the complementarity principle, the ingoing observer thinks that only the in-going matter in Fig. 2\(a\) is responsible for the matter contribution to the space-time curvature, while the Hawking radiation is essentially invisible. In Fig. 2\(b\), space-time is shown as it is recorded by the late outside observer: *only* the Hawking particles are responsible for the space-time curvature, while all in-going matter that formed the black hole in turn is invisible to him\(^{1}\). Naturally, one asks how these differences can be

\(^{1}\)This picture differs somewhat from the older versions of “complementarity”, where the outside observer does see the matter going in; we presently take the viewpoint that if an observer registers the effect of Hawking radiation on the metric, all ingoing matter becomes invisible, which is the exact time-reverse
Figure 2: a) The in-configuration of a black hole; b) the out-configuration. Dotted lines are the horizons in both pictures. In the case both the in-going matter and the out-going matter forms a single shell, there is a region at the center that both pictures seem to have in common (c).

implemented in one all-embracing theory. Are there transformations from one picture to the other? How complex may these transformations be? How can these be reconciled with causality?

5. Causality and locality

We now insist that the complementarity transformation that relates Fig. 2a with Fig. 2b should be a local transformation of dynamical physical degrees of freedom, and furthermore, that this transformation preserve the causal order of events, so that an evolution law obeying obvious causality can be phrased in either one of the two systems. This necessarily means that the light cones should not be affected by this transformation.

The metric $g_{\mu\nu}(x)$ itself, however, may be observer dependent. This is a new idea, and it appears to make some sense. Let us write

$$g_{\mu\nu}(x) = \omega^2(x) \hat{g}_{\mu\nu}(x) , \quad \text{where}$$

$$\det(\hat{g}_{\mu\nu}) = -1 , \quad \omega = (-\det(g_{\mu\nu}))^{1/8} .$$

Since the light cone is defined by solutions of the equation

$$ds^2 = g_{\mu\nu}dx^\mu dx^\nu = 0 ,$$

it is $\hat{g}_{\mu\nu}$ that determines the positions of light cones, whereas $\omega(x)$ defines the scales for rulers and clocks in the theory. Therefore, we assume that $\hat{g}_{\mu\nu}$ is observer independent, but $\omega(x)$ is not.

In Eq. (5.1), the field $\omega(x)$ has the dimension of a length. This is chosen for future convenience (the linearity of Eq. (7.4)).

of what the ingoing observer sees.
The value of $\omega(x)$ depends on whether the point $x$ is seen through a curtain of Hawking particles or directly. Thus, according to the complementarity principle, the physical events taking place at some space-time point $x$ should not explicitly depend on $\omega$. We end up with a picture where $\omega(x)$ is entirely unobservable locally, much like a local gauge parameter $\Lambda(x)$ in a gauge theory.

The fields $\hat{g}_{\mu\nu}(x)$ must obey equations of their own. Formulating these equations will be difficult, and we postpone this to future work. By first studying the physical consequences of this idea, we must cultivate some feeling about how to proceed.

At this point, we can already identify some of these consequences. Suppose we have a space-time where we are free to choose $\omega$. Suppose that cosmic censorship\cite{10}\cite{11} holds. This means that, in terms of the conventional metric $g_{\mu\nu}$, any naked singularity is hidden behind a horizon. Then, whenever we encounter such a singularity, we can adjust $\omega$ in such a way that a clock shows infinite time when approaching this singularity, by applying the transformation (3.1). Therefore, it now occurs at $t = \infty$. In practical examples, the singularity will also have been smeared out so that it effectively disappears, and then also the horizon disappears, as we will demonstrate.

6. Conformal transformations

In the case described in Section 4, where both in- and out-going matter forms a single shell, there is an internal flat region that both descriptions have in common. Since it is flat, they have the same, flat value $\hat{g}_{\mu\nu}(x) = \eta_{\mu\nu}$ for the metric. So here, the only choice we have for the transformation is the value for the scales $\omega(x)$. As is well-known, the only transformations that leave $\hat{g}_{\mu\nu} = \eta_{\mu\nu}$ intact, are the conformal transformations. If in both Figures 2a and 2b, we choose the tip where the singularity starts as the origin of space-time, then this mapping is

$$x^\mu_{\text{in}} = (-1)^{\delta^\mu_\nu} \lambda^2 \frac{x^\mu_{\text{out}}}{|(x_{\text{out}})^2|}, \quad (6.1)$$

where $(x_{\text{out}})^2$ is of course the Lorentz invariant square of the coordinate $x_{\text{out}}$. We chose the sign here such that the timelike causal order is preserved. $\lambda$ is a free parameter with the scale of a length. It will probably be close to the Planck scale. All vectors in the relevant region are timelike with respect to the origin chosen.

The transformation (6.1) is illustrated in Figure 3, where we see that indeed lightlike geodesics stay lightlike. We see cosmic censorship at work: the singularities in both frames are moved to $t \to \pm\infty$ in the other frames. As for the region outside the collapsing and the expanding shells, we might decide to choose no change in $\omega$. At the lightlike shells themselves, the mapping is singular, but this may be attributed to the fact that we are dealing with macroscopic black holes in this example, so that, in both coordinate frames, matter tends to take divergent values for their stress-energy-momentum tensors. In microscopic black holes, the transformation may be less singular at these regions.
7. Scale invariance and the stress-energy-momentum tensor

The above considerations imply that the scale factor $\omega(x)$ is ambiguous, yet it is needed to describe clocks and rulers in the macroscopic world, and it is also needed if we wish to compute the Riemann and the Ricci curvature, because they depend on the entire metric $g_{\mu\nu}$, not just $\hat{g}_{\mu\nu}$. Clearly, the world that is familiar to us is not scale invariant. Without $\omega$ we cannot define distances and we cannot define matter, but we can define the geometry of the light cones, and, if $\hat{g}_{\mu\nu}$ is non-trivial, it should describe some of the physical phenomena that are taking place. It should be possible to write down equations for it without directly referring to $\omega$.

To what extent will $\omega(x)$ be determined by $\hat{g}_{\mu\nu}(x)$, if we impose some physical conditions? For instance, we can impose the condition that the space-time singularities are all moved to $t \to \infty$, so that the interiors of black holes at the moment of collapse, are re-interpreted as the exteriors of decaying black holes, as in Figs. 2 and 3. After that is done, conformal transformations are no longer possible, because they would move infinities from the boundary back to finite points in space-time, which we do not allow. But we do wish to impose some extra “gauge” condition on $\omega(x)$, so as to fix its value, allowing us to define what clocks, rulers, and matter are.

A natural thing to impose is that all particles appear to be as light as possible. If we would have only non-interacting, massless particles, then $T^\mu_\mu = 0$. Therefore, a natural gauge to choose is the vanishing of the induced Ricci scalar:

$$ R(\omega^2 \hat{g}_{\mu\nu}) = 0 \ .$$

(7.1)
When $\hat{g}_{\mu\nu}(x)$ is given, we can compute (in 4 space-time dimensions):

$$R_{\mu\nu} = \hat{R}_{\mu\nu} + \omega^{-2} (4 \partial_{\mu} \omega \partial_{\nu} \omega - \hat{g}_{\mu\nu} \hat{g}^{\alpha\beta} \partial_{\alpha} \omega \partial_{\beta} \omega)$$

$$+ \omega^{-1} (-2 D_{\mu} \partial_{\nu} \omega - \hat{g}_{\mu\nu} \hat{g}^{\alpha\beta} D_{\alpha} \partial_{\beta} \omega) ;$$  

(7.2)

$$\omega^2 R = \hat{R} - 6 \omega^{-1} \hat{g}^{\mu\nu} D_{\mu} \partial_{\nu} \omega .$$  

(7.3)

Here, we write $R_{\mu\nu}(\omega^2 \hat{g}) = R_{\mu\nu}$, and $R_{\mu\nu}(\hat{g}_{\mu\nu}) = \hat{R}_{\mu\nu}$, and similarly the Ricci scalars. The covariant derivative $D_{\mu}$ treats the field $\omega$ as a scalar with respect to the metric $\hat{g}_{\mu\nu}$.

The condition that Eq. (7.3) vanishes gives us the equation

$$\hat{g}^{\mu\nu} D_{\mu} \partial_{\nu} \omega(x) = \frac{1}{6} \hat{R}(x) .$$  

(7.4)

This happens to be a linear equation for $\omega(x)$, which is the reason for having included the square in its definition, in Eq. (5.1), since $\omega$ is not restricted to be infinitesimal.

Now let $\omega(x)$ obey Eq. (7.4) in a region where $\hat{g}_{\mu\nu} = \eta_{\mu\nu}$, so that $\hat{R} = 0$, and consider the Fourier transform $\omega(k)$ of $\omega(x)$, which, at $k \neq 0$, we now take to be infinitesimal. The wave vector $k$ obeys $k^2 = 0$. Let this wave vector be in the $+$ direction. Then, according to Eq. (7.2), the only non-vanishing component of the Ricci tensor, and hence also the only non-vanishing component of the stress-energy-momentum tensor, is the $++$ component. Thus, we have shells of massless particles traveling in the $x^+$-direction. Adding all possible Fourier components, we see that, in the infinitesimal case, our transition to non-trivial $\omega(x)$ values leads us from the vacuum configuration to the case that massless particles are flying around. As long as $\omega(k)$ at $k \neq 0$ stays infinitesimal, these massless particles are non-interacting.

In the case of an evaporating black hole, these massless particles are the Hawking particles. Our complementarity transformation, the transformation that modifies the values of $\omega(x)$ while keeping Eq. (7.4) valid, switches on and off the effects these Hawking particles have on the metric.

Often, it will seem to be more convenient to impose that we have flat space-time, i.e. $\omega^2 \hat{g}_{\mu\nu} = \eta_{\mu\nu}$ at infinity, rather than imposing $R = 0$ everywhere. In the case of an implosion followed by an evaporating black hole (for simplicity emitting a single shell of matter as in Section 4), this would allow us to glue the different conformal regions together, to obtain Figure 4. This then gives us a region of large Ricci scalar curvature near the horizon. Ricci scalar curvature is associated with a large burst of pressure in the matter stress-energy-momentum tensor. This means that an observer who uses this frame sees a large explosion near the horizon that sends the ingoing material back out, that is, the black hole explodes classically. The local observer would be wondering what causes this “unnatural” explosion, which, at the very last moment, avoided the formation of a permanent black hole; in our theory, however, the local observer would see no reason to use this function $\omega$, so she would not notice any causality violation. Only the distant,
Figure 4: Combining the different conformal frames into one metric for the imploding and subsequently evaporating black hole, gives us a region of large Ricci scalar values $R$ near the horizon (here, the two circles). Note that the metric outside the collapsing and evaporating shells is Schwarzschild, while inside the shells it is flat. Demanding $\omega$ to be continuous across the shells implies that the $r$ coordinate should be matched there (see Ref. [16]).

outside observer would use this $\omega$, concluding that, indeed, the black hole is not an eternal one since it evaporates.

The metric described in Figure 4 is related to the metrics described in Ref. [16]. There, care was taken that no conical singularity should arise when ingoing and outgoing shells meet. Here, we do allow this singularity, interpreting it as a region of large $R$ values, which we now demand not to be locally observable.

8. Quantum Mechanics

How does one quantize a theory where the scale component $\omega$ of the metric is locally invisible? An action that is invariant under local scale transformations does exist:

$$S = C \sqrt{-g g^{\mu\gamma} g^{\nu\delta} W^\alpha_{\beta\mu\nu} W^\beta_{\alpha\gamma\delta}} ,$$

(8.1)

where $W^\alpha_{\beta\mu\nu}$ is the Weyl curvature, which is the Riemann curvature $R^\alpha_{\beta\mu\nu}$ with all trace parts removed, so that all contractions of $W$ are zero. It is the set of curvature components that is orthogonal to the contracted Ricci curvature. One can easily check that this tensor $W$ is independent of $\omega$, so that $\hat{g}_{\mu\nu}(x)$ is all that is needed to compute $W$. In four dimensions, indeed $\omega$ also cancels out in the remainder of Eq. (8.1).

What is nice about this action is that it appears to be renormalizable. By construction, of course, it is scale invariant, so the coupling parameter (the inverse of $C$ in Eq. (8.1)) is dimensionless. Matter fields that are conformally invariant also exist, for instance
\( N = 4 \) super Yang-Mills theories. However, the action (8.1) violates unitarity in a big way. The propagators will have poles of the form \( 1/(k^2 - i\epsilon)^2 \), which in general points to unphysical quantum states. Attempts by the author to formulate any quantum theory with such propagators failed.

More promising perhaps is the “primordial” or “primitive” quantization procedure proposed by the author some time ago[17][18]. Imagine some set of classical dynamical equations of motion. One could think of the classical equations associated to the action (8.1), but we can also imagine completely classical objects in the form of particles, strings or branes following classical equations of motion. One then may be able to describe the classical space of all states, and subsequently promote each and every one of these states as an element of the basis of a Hilbert space. The time evolution of these states is then generated by an operator that we will call a Hamiltonian. If the original world that we wanted to describe was strictly continuous, the number of states in Hilbert space is non denumerable, and this may cause difficulties in pursuing this program. Perhaps a fundamentally discrete model also exists, which would make our work easier. The Hamiltonian can then be constructed systematically, and its positivity and its symmetry structures can be studied.

Quantum theories of this sort could be interpreted as “hidden variable theories”. The question then arises how to deal with Bell’s inequalities. These are inequalities that describe boundaries for measurements of physical features such as spin of quantum entangled objects. If the system is a classical one, the boundaries cannot be surpassed, whereas they are surpassed in a quantum theory. For many investigators, this is sufficient reason to categorically reject all hidden variable theories. However, since the argument is a complicated one, there might be loopholes. One possible loophole is as follows.

We argued that classical systems can be rearranged such that their evolution laws are described by a quantum Hamiltonian. The eigenstates of this Hamiltonian will always be quantum superpositions. If we limit ourselves to low energy modes only, this means that, from the start, we only talk of quantum entangled states. Even if the measured features of a system under study are classical and not entangled, the invisible hidden variables will be highly entangled in the quantum mechanical sense. They should not be handled as if the were flags to be attached to particles with spin, or anything like that, but rather as an infinite sea of entangled objects. The fact that the universe appears to be in a quantum entangled state of its hidden variables can be attributed to its initial state, right at the Big Bang. It is quite conceivable that these entangled states are necessary only if one wants to describe the entire universe as being composed of tiny bits and pieces that are mutually nearly independent. The Hamiltonian is then the integral of a Hamiltonian density. This Hamiltonian density describes the local evolution law, but even for the global evolution law, we only limit ourselves to the lowest energy eigenstates, which are entangled. Thus, the emergence of entangled states for the hidden variables may be unavoidable.

Note that this does not completely resolve the issue with the Bell inequalities, but does show that they will be a lot harder to use as a “no go” theorem for hidden variables that usually assumed.
9. The Baker-Campbell-Hausdorff expansion

According to the theory just described, we have discrete sets of physical data that evolve according to non-quantum mechanical, deterministic laws. A common version of this theory is one where also the time evolution is fundamentally discrete. In that case, the theory can be defined in terms of an evolution operator $U_0$ that describes one step in time. Hence, we would like to write

$$U_0 = e^{-iH},$$

(9.1)

where $U_0$ acts on the states $|n\rangle$ that are treated as basis vectors of a Hilbert space, so that $H$ plays the role of a *quantum* Hamiltonian. We get conventional quantum mechanics if we can assure that $H$ is bounded from below,

$$\langle \psi | H | \psi \rangle \geq \langle \psi_0 | H | \psi_0 \rangle,$$

(9.2)

for one state $|\psi_0\rangle$ and all states $|\psi\rangle$, where $|\psi\rangle$ may be any superposition of states $|n\rangle$, in particular any eigen state of $H$. In principle, an operator obeying this demand is not difficult to construct. Using the Fourier transform

$$x = \pi - \sum_{k=1}^{\infty} \frac{2}{k} \sin kx,$$

(9.3)

we derive:

$$H = \pi + \sum_{n=1}^{\infty} \frac{i}{n} (U_n^0 - U^{-n}_0).$$

(9.4)

This Hamiltonian has only eigenvalues between 0 and $2\pi$, and it reproduces Eq. (9.1). One can imagine using this Hamiltonian to write our deterministic system in a quantum mechanical formalism, allowing states to be quantum mechanically entangled. The importance of having a lower bound of the Hamiltonian eigenvalues is that the lowest states can be identified as the ‘vacuum state’, and the first excited states can be interpreted as states containing particles. Thermodynamics gives us mixed states with probabilities

$$\rho = Ce^{-E/kT},$$

(9.5)

However, in *extensive* systems, such as Fock space for a quantum field theory, this Hamiltonian is not good enough, for two (related) reasons. One is that the very high $k$ contributions in Eq. (9.4) refer to large times, and this implies that these contributions are non-local. If interactions spread with the speed of light, the Hamiltonian will generate direct interactions over spatial distances proportional to $k$. This necessitates a cut-off: if the time steps are assumed to be one Planck time, then also $k$ time steps should be kept short enough so as to reproduce local physics at the Standard Model scale. Using a cutoff in Eq. (9.4) gives an energy spectrum as sketched in Fig. 5.

In this figure, a smooth cut-off has been applied (cutting off the large $k$ values with a Gaussian exponential). We see that, as a consequence, the lowest energy states are
severely affected; their energy eigenvalues are now quadratic in the momenta $k$, so that, here, the Hamiltonian does not reproduce the correct evolution operator (9.1). This region, however, is important when applying the laws of thermodynamics, since these states dominate in the Boltzmann expression (9.5).

The second problem is that we expect a Hamiltonian to decouple when states are considered that are spatially separated: $H = H_1 + H_2$, but then one cannot maintain that the eigenvalues of the entire Hamiltonian stay within the bounds $(0, 2\pi)$.

The construction of an extensive Hamiltonian was suggested in [18]. Consider a cellular automaton, of which the evolution laws is defined in two steps:

$$U_0 = AB, \quad A = \prod_{\vec{x}} A(\vec{x}) \quad B = \prod_{\vec{x}} B(\vec{x}).$$

(9.6)

We assume all $A$ operators to produce locally independent evolutions:

$$[A(\vec{x}), A(\vec{x}')] = 0,$$

(9.7)

and similarly the $B$ operators:

$$[B(\vec{x}), B(\vec{x}')] = 0,$$

(9.8)

but the $A$ evolution does not commute with the $B$ evolution, when neighboring points $\vec{x}, \vec{x}'$ are concerned:

$$[A(\vec{x}), B(\vec{x}')] = 0 \quad \text{only if} \quad |\vec{x}' - \vec{x}| > 1.$$  

(9.9)

We can now use the finite fourier transforms to define operators $a(\vec{x})$ and $b(\vec{x})$:

$$A = e^{-ia}, \quad a = \sum_{\vec{x}} a(\vec{x}) \quad B = e^{-ib} \quad b = \sum_{\vec{x}} b(\vec{x}),$$

(9.10)

where $a(\vec{x})$ and $b(\vec{x})$ obey the same commutation restrictions (9.7), (9.8) and (9.9). Now, we can use the Baker-Campbell-Hausdorff expansion to write $U_0$ as an exponential:

$$U_0 = e^{-iH}, \quad H = a + b - \frac{1}{2i}[a, b] + \frac{1}{12}([a, [a, b]] + [b, [a, b]]) - \frac{1}{24}[b, [a, [a, b]]] + \cdots,$$

(9.11)
which continues as an infinite series that is exclusively built from commutators.

The importance of this expression is that each of these commutators are non vanishing only if they consist of neighboring operators; therefore, this Hamiltonian can be written as the sum of Hamilton densities. Therefore, it seems as if the Hamiltonian is constructed just as in a quantized field theory. Distant parts of this ‘universe’ evolve independently. The total Hamiltonian has eigenvalues much greater than $2\pi$, and are all bounded from below. If now we concentrate on the lowest lying states, we see that these consist of localized particles resembling what we see in the real world. They have positive energies, so that thermodynamics applies to them.

Unfortunately, there is a caveat: the BCH series (9.11) does not converge. To be precise, the operator $H$ is ambiguous when two of its eigenvalues get further separated than $2\pi$, as can be seen directly from its definition (9.1). That is where the series’ radius of convergence ends.

One must deduce that the series (9.11) cannot be truncated easily; it will have to be resummed carefully, and whether or not a resummation exists remains questionable. We can argue that these difficulties only refer to matrix elements between states whose energy eigenvalues are more than the Planck energy apart, which is a domain of quantum physics that has never been addressed experimentally anyway, so maybe they can safely be ignored. Yet this situation is unsatisfactory, so more work is needed.

10. Conclusions

There are reasons to suspect that the presence or absence of Hawking particles, and the effects they have on the space-time metric, are observer dependent. This would mean that at least the scale component $\omega(\vec{x}, t)$ of the metric is not locally observable. The scale of things can only be observed if we assume the space and time surrounding us to be sufficiently void of too large amounts of matter such as Hawking radiation; only then can we compare the scale of things at different locations in space and time.

If this is so then quantum mechanics itself has to be reformulated when the gravitational force is involved. In the last two sections, we sketched a possible route towards a quantum theory. Though not free of problems, what we may have illustrated here is that quantum statistical features may originate in a deterministic dynamical theory at the Planck scale.

References


