Spectral dimensions from the spectral action

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Motivation

Is there spontaneous Dimensional Reduction in Short-Distance Quantum Gravity?

Important question in Quantum Gravity: Structure of space-time at very short distances?

When lowering the distance scale / increasing the energy scale

- number of dimensions increases (as e.g. in the ADD model)?
- number of dimensions decreases (as e.g. in QEG or CDT)?
- volumes and areas become quantised (as e.g. in QLG)?
- space-time becomes discrete (as in the work presented here)?
- something completely else happens?
Fractal-like properties of a space(-time):

- Haussdorff dimension
  - Determined by $\# N$ of balls necessary to cover a set of points:
    $$N(R) \propto 1/R^D$$

- Real line: $N(R) \propto 1/R$, i.e., $D = 1$
  - Coast of England: $D \approx 1.2$

- Spectral dimension $D_s$
  - Consider diffusion of scalar test particle on some manifold
  - Average return probability $\mathcal{P}$ “feels” space-time dimension
    - $\mathcal{P} \propto T^{-D_s/2}$ with (fictitious) diffusion time $T$

- Spectral dimension = Hausdorff dim. of momentum space
Heuristic picture for the concept of a spectral dimension:

- Ping-pong ball sees three dimensions 😊
- Table-football ball sees two dimensions 😕?

→ Interactions change the spectral dimension 😊!
Spectral dimension near and beyond Planck scale:

Several ansätze for Quantum Gravity suggest $D_s = 2$!


Proposed physical picture:

Focusing of geodesics (Asymptotic Silence)
Motivation

Is the value $D_S = 2$ in the UV generic for all approaches to Quantum Gravity?

Here:
Calculate the (classical) spectral dimension for Connes’ non-commutative (resp., almost commutative) geometry at short distances.
**Generalised Spectral Dimensions**

**Diffusion on classical manifold with metric $g$:**

- Characterised by probability density $P(x, x'; T)$
- Average return prob. $\mathcal{P}(T) = \frac{1}{V} \int d^d x \sqrt{g(x)} P(x, x' = x; T)$
- Spectral dimension $D_S(T) = -2 T \frac{\partial}{\partial T} \ln \mathcal{P}(T)$
- Fictitious diffusion time $T$: Resolution scale $\mu \propto 1/\sqrt{T}$

**Calculation from 2-point function (≈ inverse propagator)**

- (Generalised) Laplacian $D^2$: $\partial_T P(x, x'; T) = D^2 P(x, x'; T)$
- For kinetic term $F(D^2)$: $\partial_T P(x, x'; T) = F(D^2) P(x, x'; T)$
- Average return probability $\mathcal{P}(T) = const. \int_0^\infty dz \ z \ e^{-TF(z)}$
- Spectral dimension $D_S(T) = -2 T \frac{\mathcal{P}'(T)}{\mathcal{P}(T)}$ with $\mathcal{P}'(T) = \partial_T \mathcal{P}(T)$
Connes’ non-commutative / almost commutative geometry:

- Spectral triple \( \{A, \mathcal{H}, D\} \): Algebra, Hilbert space, Dirac operator
- Continuous spectral triple \( \otimes \) discrete spectral triple (\( \cong \) Riemannian manifold) (non-comm. part \( \rightarrow \) gauge sym.)

1. Generalisation of Riemannian geometry
2. Universal formula for the action of elementary fields
3. “Generating Functional” for Standard Model (SM) coupled to gravity at low scales (geometrical derivation of SM)
4. Very high energies: Framework for unification of SM & gravity
Spectral action for bosons:

\[ S_{\chi,\Lambda} = \text{Tr}(\chi(D^2/\Lambda^2)) \]

1. \( \chi \) positive function
2. \( D \) Dirac operator: contains spin 0, 1 & 2 fields
3. \( \Lambda \) physical scale, e.g., Planck mass or GUT Scale

Conjecture:
“High energy bosons do not propagate”
...based on qualitative arguments...
Dirac operator:

\[ D^2 = -(\nabla^2 + E) \]

with

\[ E = -i\gamma^\mu\gamma_5 \nabla_\mu \phi - \phi^2 - \frac{1}{4} R + \frac{i}{4}[\gamma^\mu, \gamma^\nu]F_{\mu\nu}. \]

1. \( \phi \) scalar field,
   \( F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \) (Abelian) gauge field,
   \( R \) Ricci scalar
2. \( \nabla_\mu = \nabla_\mu^{LC} + iA_\mu \) Levi-Cevita spin connection & gauge potential
3. Curvature \( \Omega_{\mu\nu} := [\nabla_\mu, \nabla_\nu] = -\frac{1}{4} \gamma^\rho \gamma^\sigma R_{\rho\sigma\mu\nu} + iF_{\mu\nu} \)
Expand Spectral Action for $\chi(z) = e^{-z}$ to 2nd order in fields:

$$S^{(2)} \propto \int d^4x \left( \phi G_0(-t\partial^2)\phi + A_\mu G_1(-t\partial^2)A_\mu + h_{\mu\nu} G_2(-t\partial^2)h_{\mu\nu} \right)$$

These functions are non-polynomial in $z = tp^2 = p^2/\Lambda^2!$

$G_s(z)$

- $G_0(z = tp^2)$ scalar,
- $G_1(z)$ gauge field,
- $G_2(z)$ graviton.

$$\lim_{z \to \infty} G_{0,1} = 0$$
$$\lim_{z \to \infty} G_2 = -3/2$$

Note that for scalar, i.e., Higgs, $F_0(0) = m^2 = -\Lambda^2/(4\pi)^2 < 0. \implies \text{SSB!}$
Effective Field Theory:
Expand in $t = 1/\Lambda^2$ to generate action polynomial in derivatives and **truncate** at some fixed order in $-t \partial^2$, resp., $p^2/\Lambda^2$:

$G_s(z)$

$G_s(z)$ at $O(z^3)$ (thick lines)

$G_s(z)$ at $O(z)$ (thin lines)
Effective Action for a generic function $\chi$ in $S_{\chi,\Lambda} = \text{Tr}(\chi(D^2/\Lambda^2))$, part quadratic in fields:

\[
S_{\chi,\Lambda}^{(2)} = \frac{\Lambda^2}{(4\pi)^2} \int d^4x \left[ \phi \mathcal{F}_{0,\chi} \left(-\partial^2/\Lambda^2\right) \phi + A_\mu \mathcal{F}_{1,\chi} \left(-\partial^2/\Lambda^2\right) A_\mu + h_{\mu\nu} \mathcal{F}_{2,\chi} \left(-\partial^2/\Lambda^2\right) h_{\mu\nu} \right].
\]

\[
\mathcal{F}_{0,\chi}(z) = -Q_1 + Q_0 \frac{z^2}{2} - Q_{-1} \frac{z^2}{12} + Q_{-2} \frac{z^3}{120} + \ldots
\]

\[
\mathcal{F}_{1,\chi}(z) = Q_0 \frac{4z}{3} - Q_{-1} \frac{4z^2}{15} + Q_{-2} \frac{z^3}{35} + \ldots
\]

\[
\mathcal{F}_{2,\chi}(z) = -Q_2 + Q_1 \frac{z}{12} - Q_0 \frac{z^2}{40} + Q_{-1} \frac{z^3}{336} + \ldots
\]

$\chi \mapsto Q_n$ (unique, $Q_n = 1$ for $\chi = e^z$)

$Q_n$’s cannot be adjusted independently!
**Results**

\[ D_s(T) \]

- \( F_0 \) scalar,
- \( F_1 \) gauge field,
- \( F_2 \) graviton.

**IR** \( (T \to \infty) \):
\[ D_S \to 4 \]

**UV** \( (T \to 0) \):
\[ D_S(0) = \frac{4}{N_{\text{max}}} \]
determined by highest power of truncation!

\[ D_S(0) \to 0 \text{ for } N_{\text{max}} \to \infty \]
Include full momentum dependence of inverse propagators:

- Regularise mom. integrals $\mathcal{P}(T; \Lambda_{\text{UV}}) = \int_{\Lambda_{\text{UV}}}^{\Lambda_{\text{UV}}} \frac{d^4p}{(2\pi)^4} e^{-TF(p^2)}$
- $\Lambda_{\text{UV}} \to \infty$ (NB: $\Lambda$ fixed!)
- “Late-time” expansion of functions: one can analytically show

$$D_S(T) = \lim_{\Lambda_{\text{UV}} \to \infty} D_S(T; \Lambda_{\text{UV}}) = 0.$$ 

Generic function $\chi$ in Spectral Action $\implies$ Nonanalytic inv. prop.

**Consequence:**

Spectral dimensions vanish in UV / for very small distances!
Space-time fractures into non-communicating points!
Almost-commutative geometry:

- Calculation of the scale-dependent spectral dimensions of bosons from a spectral action.
- **Spectral dimensions vanish beyond Planck scale!** ("High-energy bosons do not propagate!")
- **Space-time fractures into non-communicating points!**

? Different than in all other approaches to Quantum Gravity!

? **Spectral dimensions if quantum propagators are used?**