Solving the sign problems of the massless lattice Schwinger model

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Complex action problem

- Path integral for a quantum field theory:

$$Z = \int D[\Phi] \, e^{-S[\Phi]}$$

- In a Monte Carlo calculation we generate field configurations with distribution

$$P[\Phi] \propto e^{-S[\Phi]}$$

- In general lattice field theories with finite chemical potential μ or a topological term have actions S with an imaginary part, such that

$$e^{-S[\Phi]} \in \mathbb{C}$$

- The Boltzmann factor has a complex phase and cannot be used as a probability weight.

"Complex action problem" or "Sign problem"

- Generic feature of finite density field theories both, on the lattice and in the continuum, for bosonic and fermionic theories.

Solving the complex action problem with dual variables

- One can use different variables to formulate a quantum field theory.

- Maybe it is possible to identify alternative variables such that the partition sum has only real and positive factors?

- For several systems such an alternative representation without sign problem was found.

- These "dual degrees of freedom" are discretized loops for matter fields and discretized surfaces for gauge fields:

$$Z = \sum_{\{l,s\}} W[l,s] \quad , \quad W[l,s] \in \mathbb{R}$$

- The Monte Carlo calculation can be done in terms of the new variables.

- Dual variables sometimes give new insight into physical mechanisms.

Examples of complex action problems solved with dual variables

- $\mathbb{Z}_3$ spin model with chemical potential. (PRL 2011, CPC 2012)
- Polyakov loop model with chemical potential (SU(3) spin model). (NPB 2011& 2012)
- $\mathbb{Z}_3$ gauge-Higgs model with chemical potential. (PRD 2012, CPC 2013)
- Charged ϕ^4 field at finite density (relativistic Bose gas). (NPB 2013, PLB 2013)
- $U(1)$ gauge-Higgs model with chemical potential. (PRL 2013, CPC 2013)
- Scalar QED$_2$ with topological term. (PoS 2014, work in preparation)
- Massless Schwinger model with topological term and chemical potential. This talk

Problem: There is no generally applicable strategy for dualization.
Each class of models has to be attacked individually.

Massless Schwinger model on the lattice

- Partition sum and lattice action: ($\mu \neq 0 \;\Rightarrow$ 2 flavors)

$$Z \;=\; \int \mathcal{D}[U]\mathcal{D}[\,\overline{\psi},\psi\,]\; e^{-\,S_G[U]\,-\,i\,\theta\,Q[U]\,-\,S_F[U,\overline{\psi},\psi]}$$

$$S_F \;=\; \frac{1}{2}\sum_{n,\nu}\gamma_\nu(n)\Big[e^{+\mu\,\delta_{\nu,2}}\,U_\nu(n)\,\overline{\psi}(n)\,\psi(n+\hat{\nu}) \;-\; e^{-\mu\,\delta_{\nu,2}}\,U_\nu(n)^*\,\overline{\psi}(n+\hat{\nu})\,\psi(n)\Big]$$

staggered sign function: $\gamma_1(n) \;=\; 1 \quad , \quad \gamma_2(n) \;=\; (-1)^{n_1}$

$$S_G[U] + i\,\theta\,Q[U] \;=\; -\frac{\beta}{2}\sum_n\left[\,U_p(n) \,+\, U_p(n)^*\,\right] + \frac{\theta}{4\pi}\sum_n\left[\,U_p(n) \,-\, U_p(n)^*\,\right]$$

$$=\; -\,\eta\,\sum_n U_p(n) \,-\, \overline{\eta}\sum_n U_p(n)^*$$

$$\eta \;=\; \frac{\beta}{2} - \frac{\theta}{4\pi} \quad , \quad \overline{\eta} \;=\; \frac{\beta}{2} + \frac{\theta}{4\pi}$$

Absorbing the staggered sign function

- For gauge groups containing -1 the staggered signs can be absorbed in the links:

$$U_\nu(n) \;\to\; \gamma_\nu(n)\, U_\nu(n) \quad , \qquad U_p(n) \;\to\; -\,U_p(n) \quad , \qquad \mathcal{D}[U] \;\to\; \mathcal{D}[U]$$

Staggered $\gamma_\nu(n)$ removed from S_F. Sign of β and θ is flipped. Works also in 4D.

- Simplified partition sum:

$$Z \;=\; \int \mathcal{D}[U]\; e^{-\,\eta \sum_n U_p(n)}\; e^{-\,\overline{\eta} \sum_n U_p(n)^*}\; Z_F[U]$$

$$Z_F[U] \;=\; \int \mathcal{D}[\,\overline{\psi}, \psi\,]\; e^{-\frac{1}{2}\sum_{n,\nu}[\,e^{+\mu\,\delta_{\nu,2}}\, U_\nu(n)\, \overline{\psi}(n)\, \psi(n+\hat{\nu}) \,-\, e^{-\mu\,\delta_{\nu,2}}\, U_\nu(n)^*\, \overline{\psi}(n+\hat{\nu})\, \psi(n)]}$$

Expanding the fermionic partition sum ($\mu = 0$ here)

- Expanding the Boltzmann factors turns Z_F into a sum of configurations of occupation numbers $k_\nu(n), \overline{k}_\nu(n) \in \{0, 1\}$:

$$Z_F[U] = \int \mathcal{D}[\overline{\psi}, \psi] \prod_{n,\nu} e^{-\frac{1}{2} U_\nu(n)\, \overline{\psi}(n)\, \psi(n+\hat{\nu})} e^{+\frac{1}{2} U_\nu(n)^*\, \overline{\psi}(n+\hat{\nu})\, \psi(n)}$$

$$= \int \mathcal{D}[\overline{\psi}, \psi] \prod_{n,\nu} \sum_{k_\nu(n), \overline{k}_\nu(n)=0}^{1} \left[-\frac{U_\nu(n)}{2}\, \overline{\psi}(n)\, \psi(n+\hat{\nu}) \right]^{k_\nu(n)} \left[\frac{U_\nu(n)^*}{2}\, \overline{\psi}(n+\hat{\nu})\, \psi(n) \right]^{\overline{k}_\nu(n)}$$

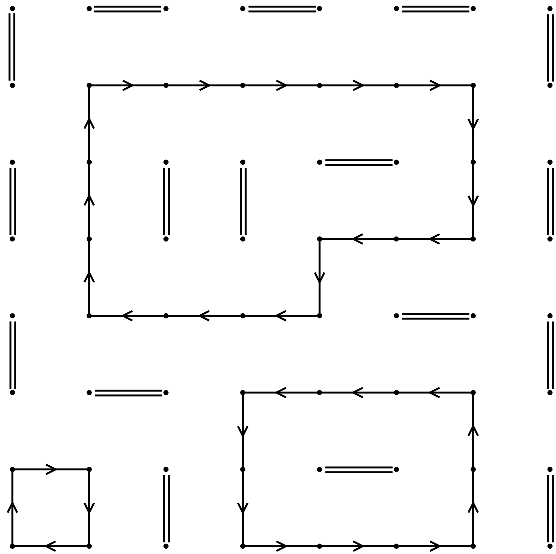
$$= \sum_{\{k, \overline{k}\}} \prod_{n,\nu} \frac{(-1)^{k_\nu(n)}}{2^{\,k_\nu(n)+\overline{k}_\nu(n)}}\, U_\nu(n)^{k_\nu(n)-\overline{k}_\nu(n)}\; S[k, \overline{k}]$$

- Remaining Grassmann integral gives zero or a sign:

$$S[k, \overline{k}] = \int \mathcal{D}[\overline{\psi}, \psi] \prod_{n,\nu} \left(\overline{\psi}(n)\, \psi(n+\hat{\nu}) \right)^{k_\nu(n)} \left(\overline{\psi}(n+\hat{\nu})\, \psi(n) \right)^{\overline{k}_\nu(n)}$$

Admissible fermion configurations

Nontrivial $S[k,\overline{k}]$ only for configurations of the $k_\nu(n), \overline{k}_\nu(n)$ where each $\psi(n)$ and $\overline{\psi}(n)$ is activated exactly once. $\Rightarrow$ Complete filling of the lattice with closed loops and dimers.



Fermionic partition sum

- The fermionic partition sum is thus a sum over configurations $\{l, d\}$ of loops and dimers:

$$Z_F[U] \;=\; \left(\frac{1}{2}\right)^V \sum_{\{l,d\}} (-1)^{N_L} \; (-1)^{\frac{1}{2}\sum_l L(l)} \; (-1)^{\sum_l W(l)} \prod_l \prod_{(n,\nu)\in l} U_\nu(n)^{s_\nu(n)}$$

N_L ..... number of loops

$L(l)$ ..... length of the loop l

$W(l)$ ..... winding number of the loop l around compact time

$s_\nu(n)$ ..... occupation number of a link in a loop ($s_\nu(n) \in \{-1, +1\}$)

- The loops are dressed with link variables which have to be integrated over. Since ...

$$\int dU_\nu(n) \; [U_\nu(n)]^j \;=\; \delta_{j,0}$$

... the link variables on the loops have to be compensated with plaquettes from the expansion of the Boltzmann factor with the gauge action.

Expansion of the Boltzmann factor of the gauge action:

- Factorization of the gauge action Boltzmann factor as product over plaquettes:

$$e^{-S_G[U]-i\theta Q[U]} = \prod_n e^{-\eta\, U_p(n)}\, e^{-\overline{\eta}\, U_p(n)^{-1}}$$

- Expansion of the individual exponentials:

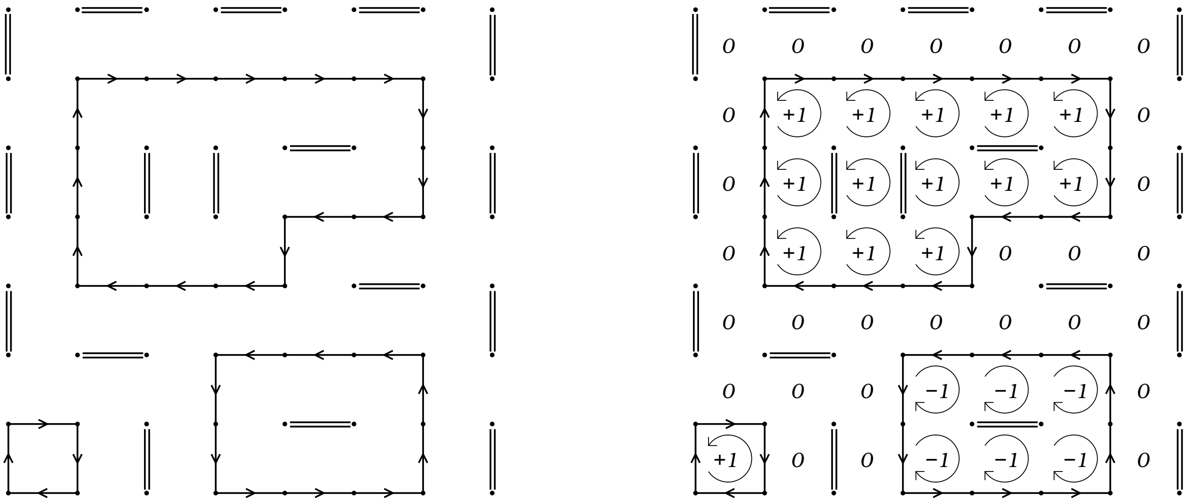
$$e^{-\eta\, U_p(n)}\, e^{-\overline{\eta}\, U_p(n)^{-1}} = \sum_{p(n)\in\mathbb{Z}} (-1)^{p(n)}\, I_{|p(n)|}\Big(\sqrt{\eta\overline{\eta}}\Big) \left(\sqrt{\frac{\eta}{\overline{\eta}}}\right)^{p(n)} U_p(n)^{p(n)}$$

$p(n) \in \mathbb{Z}$ .... plaquette occupation numbers

$I_{|p(n)|}(\sqrt{\eta\overline{\eta}}\,)$ .... modified Bessel functions

Saturation of the link variables:

Example how link variables along the loops are saturated with flux from the plaquettes:



Dual form of the partition sum:

- The partition function is a sum over all admissible configurations of loops l, dimers d and plaquette occupation numbers p:

$$Z = \left(\frac{1}{2}\right)^V \sum_{\{l,d,p\}} (-1)^{N_L + N_P + \frac{1}{2}\sum_l L(l)} \prod_n I_{|p(n)|}\left(\sqrt{\eta\overline{\eta}}\right) \left(\sqrt{\frac{\eta}{\overline{\eta}}}\right)^{p(n)}$$

Here we have introduced the total plaquette occupation number:

$$N_P = \sum_n p(n)$$

- The dual representation is an exact mapping. Left to show:

$$(-1)^{N_L + N_P + \frac{1}{2}\sum_l L(l)} = 1 \qquad \forall \quad \text{admissible configurations}$$

Sign formula for the loops:

- A factorization theorem reduces the problem to analyzing single loops.
- For a single loop l of arbitrary shape one can recursively show the relation:

$$2\,N_P(l) \;+\; 2 \;=\; 4\,N_D(l) \;+\; L(l)$$

$N_P(l)$ .... number of plaquettes inside the loop

$N_D(l)$ .... number of dimers inside the loop

$L(l)$ .... length of the loop

- Sum over all loops l:

$$\sum_l N_P(l) \;+\; \sum_l 1 \;=\; 2\sum_l N_D(l) \;+\; \frac{1}{2}\sum_l L(l)$$

- Rearrange terms:

$$N_P \;+\; N_L \;+\; \frac{1}{2}\sum_l L(l) \;=\; 2\sum_l N_D(l) \;+\; \sum_l L(l) \;=\; \text{even}$$

- $\Rightarrow$ Problem solved: $\quad (-1)^{N_L + N_P + \frac{1}{2}\sum_l L(l)} \;=\; 1$

Dual representation of the massless Schwinger model with θ-term:

- Partition function is a sum over admissible configurations of loops, dimers and plaquette occupation numbers:

$$Z = \left(\frac{1}{2}\right)^V \sum_{\{l,d,p\}} \prod_n I_{|p(n)|}\left(\sqrt{\eta\overline{\eta}}\right) \left(\sqrt{\frac{\eta}{\overline{\eta}}}\right)^{p(n)}$$

- Fermion loops that wind forward (backward) in time are weighted with $e^{+\mu N_2}$ ($e^{-\mu N_2}$)

- The partition sum has only real and positive terms for positive $\eta = \frac{\beta}{2} - \frac{\theta}{4\pi}$ and $\overline{\eta} = \frac{\beta}{2} + \frac{\theta}{4\pi}$.

- Monte Carlo calculations can be done directly in terms of the dual variables.

- Bulk observables are obtained as derivatives with respect to the parameters. For correlators one can introduce local sources.

Summary

- Rewriting lattice field theories to dual variables can solve the complex action problem.

- Two sources of complex action in the Schwinger model: chemical potential and topological term.

- For the massless case the partition sum can be mapped to a dual representation with only real and positive terms.

- The dual variables are fermion loops and dimers and the corresponding plaquette occupation numbers.

- Positivity of the weights comes from interplay of fermion properties **and** the gauge interaction.

- First example of a complete (large β, $\mu \neq 0$, $\theta \neq 0$) positive dualization of fermions interacting with a gauge field.