All Order Linearized Hydrodynamics from Fluid/Gravity Correspondence

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following some older works with Edward Shuryak
Generalize NS hydro by introducing ALL order dissipative terms in the gradient expansion of fluid stress tensor

\[(\nabla \nabla u) \text{ we keep} \quad (\nabla u)^2 \text{ we neglect}\]

Extract momenta-dependent viscosity function \(\eta(\omega, q)\) by matching two-point correlation functions of the stress tensor with correlation functions computed from BH AdS/CFT (fluid/gravity correspondence).

We have set the problem, but at the time failed to solve it completely. (We have done it now!)

We have done phenomenological studies of the effects of all-order gradients on entropy/multiplicity production in HI collisions

**Motivation:** Experiments (RHIC,LHC) probe systems with finite gradients. Phenomenologically observed low viscosity is an “effective” viscosity measured at momentum typical for process in study.

High order gradients are very big in early stages of HI collisions

Small perturbations/correlations on top of global explosion are sensitive to high gradients. This is where our results are most applicable

Relativistic Navier-Stokes hydrodynamics is non-causal/non-stable. Causality is supposed to be restored after summation of all orders
Relativistic Hydrodynamics

Energy momentum tensor

\[ \langle T^{\mu\nu} \rangle = (\epsilon + P) u^\mu u^\nu + P g^{\mu\nu} + \Pi^{\mu\nu} \]

\[ u_\nu = -1/\sqrt{1-\beta^2}, \quad u_i = \beta_i/\sqrt{1-\beta^2} \]

\[ \Pi^{\mu\nu} - \text{tensor of dissipations (ideal fluid: } \Pi^{\mu\nu} = 0) \]

Landau frame choice: \[ u_\mu \Pi^{\mu\nu} = 0. \]

Navier Stokes hydro (expanding up to first order in the velocity gradient)

\[ \Pi_{ij} = -\eta_0 \sigma_{ij}, \quad \sigma_{ij} = \frac{1}{2} \left( \partial_i \beta_j + \partial_j \beta_i - \frac{2}{3} \delta_{ij} \partial \beta \right), \quad \Pi_{vv} = \Pi_{vj} = 0. \]

\[ \nabla_\mu \langle T^{\mu\nu} \rangle = 0 \quad \rightarrow \quad \text{Navier – Stokes Eqns.} \]
Shuryak and M. L.: Introduce all order gradient expansion of $\langle T^{\mu\nu} \rangle$:

$$\Pi_{ij} = - \left[ \eta(\omega, q^2) \sigma_{ij} + \zeta(\omega, q^2) \pi_{ij} \right],$$

where $\pi_{ij}$ is a third order tensor structure

$$\pi_{ij} = \partial_i \partial_j \partial \beta - \frac{1}{3} \delta_{ij} \partial^2 \partial \beta$$

$\eta = \eta[\nabla^2, (u \nabla)]; \quad \zeta = \zeta[\nabla^2, (u \nabla)];$

$$\nabla^2 \rightarrow \omega^2 - q^2 \text{ and } (u \nabla) \rightarrow -i \omega.$$

We keep the nonlinear dispersion to all orders, but

We neglect nonlinear interactions (though some terms could be recovered).
Results: Viscosities from the Fluid/Gravity correspondence

Analytical results in the hydrodynamic regime \( \omega, q \ll 1 \quad (\pi T = 1) \):

\[
\eta(\omega, q^2) = 2 + (2 - \ln 2)i\omega - \frac{1}{4}q^2 - \frac{1}{24} \left[ 6\pi - \pi^2 + 12 \left( 2 - 3\ln 2 + \ln^2 2 \right) \right] \omega^2 + \cdots
\]

\[
\zeta(\omega, q^2) = \frac{1}{12} \left( 5 - \pi - 2\ln 2 \right) + \cdots
\]

Blue terms are new!

Modified sound dispersion:

\[
\omega = \pm \frac{1}{\sqrt{3}}q - \frac{i}{6}q^2 \pm \frac{1}{24\sqrt{3}} \left( 2\ln 2 - 3 \right) q^3 + \frac{i}{288} \left( 8 - \frac{\pi^2}{3} + 4\ln^2 2 - 4\ln 2 \right) q^4 +
\]
Real parts of the viscosities are decreasing functions of momenta. Oscillations are consistent with the expectations about the viscosities have infinitely many complex poles.

Imaginary parts have a clear maximum near $\omega \sim 2$, introducing a (new?) transition scale.

Viscosity vanish at large momenta, which is probably what is required to restore causality.

$\zeta$ is always subleading vs $\eta$. 

$q^2 = 0$
AdS/CFT correspondence: weakly coupled super-gravity in $AdS_5 \times S_5$ is “dual” to strongly coupled $\mathcal{N} = 4$ SYM gauge theory in 4d
All dissipative effects take place at the horizon. There is no dissipation in the bulk. Gravitons propagate signals from the horizon to the boundary, where the hologram is captured. The bulk acts as a highly nonlinear dispersive medium.

First approach: Perturb metric near horizon (membrane paradigm) and read off the response of the system at the boundary.

Alternative approach: send a signal from one point at the boundary to another (two point correlations of stress energy tensor).
5d GR with negative cosmological constant:

\[ S = \frac{1}{16\pi G_N} \int d^5 x \sqrt{-g} \left( R + 12 \right), \]

Einstein Equations

\[ E_{MN} \equiv R_{MN} - \frac{1}{2} g_{MN} R - 6 g_{MN} = 0. \]

Solution: Boosted Black Brane in asymptotic AdS$_5$

\[ ds^2 = -2u_\mu dx^\mu dr - r^2 f(br) u_\mu u_\nu dx^\mu dx^\nu + r^2 P_{\mu\nu} dx^\mu dx^\nu, \]

\[ f(r) = 1 - 1/r^4 \quad \text{and} \quad P_{\mu\nu} = \eta_{\mu\nu} + u_\mu u_\nu \]

Hawking temperature

\[ T = \frac{1}{\pi b}, \]
Promote $\beta_i$ and $b$ into a slowly varying functions of boundary coordinates $x^\alpha$

$$ds^2 = -2u_\mu(x^\alpha)dx^\mu dr - r^2 f(b(x^\alpha)r) u_\mu(x^\alpha)u_\nu(x^\alpha)dx^\mu dx^\nu + r^2 P_{\mu\nu}(x^\alpha)dx^\mu dx^\nu,$$

Use gradient expansion of the fields $u(x) = u_0 + \delta x \nabla u$ and $b(x) = b + \delta x \nabla b$

to set up a perturbative procedure

The resulting energy momentum tensor

$$\langle T^{\mu\nu} \rangle = T_{\mu\nu}^{ideal} + \Pi^{\mu\nu}_{NS} + \tau_R (u \nabla) \Pi^{\mu\nu}_{NS} + O[ (\nabla u)^2 ]$$

$$\frac{\eta_0}{s} = \frac{1}{4 \pi}, \quad \tau_R = 2 - \log(2)$$
We do it somewhat differently, linearizing in the velocity amplitude

\[ u_\mu(x^\alpha) = (-1, \epsilon \beta_i(x^\alpha)) + \mathcal{O}(\epsilon^2), \quad b(x^\alpha) = b_0 + \epsilon b_1(x^\alpha) + \mathcal{O}(\epsilon^2), \]

"seed" metric, i.e., a linearized version of the BH metric

\[ ds^2_{\text{seed}} = 2dr dv - r^2 f(r) dv^2 + r^2 d\vec{x}^2 - \epsilon \left[ 2\beta_1(x^\alpha) dr dx^i + \frac{2}{r^2} \beta_i(x^\alpha) dv dx^i + \frac{4}{r^2} b_1(x^\alpha) dv^2 \right] + \mathcal{O}(\epsilon^2), \]

\[ ds^2 = ds^2_{\text{seed}} + ds^2_{\text{corr}}[\beta] \quad \text{gauge fix} \quad g_{rr} = 0, \quad g_{r\mu} \propto u_\mu \]

\[ ds^2_{\text{corr}} = \epsilon \left( -3h dr dv + \frac{k}{r^2} dv^2 + r^2 h d\vec{x}^2 + \frac{2}{r^2} j_i dv dx^i + r^2 \alpha_{ij} dx^i dx^j \right) \]

\[ h[\beta], \quad k[\beta], \quad j[\beta], \quad \alpha[\beta] \] are to be found by solving the Einstein equations.

**Boundary cond:** no singularities, no modification to AdS asymptotics at \( r \to \infty \)

\[ h < \mathcal{O}(r^0), \quad k < \mathcal{O}(r^4), \quad j_i < \mathcal{O}(r^4), \quad \alpha_{ij} < \mathcal{O}(r^0). \]
We consider a hypersurface $\Sigma$ at constant $r$.

**Vector $n_M$ normal to $\Sigma$:**

$$n_M = \frac{\nabla_M r}{\sqrt{g^{MN} \nabla_M r \nabla_N r}}.$$

**Induced metric $\gamma_{MN}$ on $\Sigma$:**

$$\gamma_{MN} = g_{MN} - n_M n_N.$$

**Extrinsic curvature tensor $K_{MN}$:**

$$K_{MN} = \frac{1}{2} \left( n^A \partial_A \gamma_{MN} + \gamma_{MA} \partial_N n^A + \gamma_{NA} \partial_M n^A \right).$$

The stress tensor for the dual fluid

$$\langle T^\mu_\nu \rangle = \lim_{r \to \infty} \tilde{T}^\mu_\nu (r); \quad \tilde{T}^\mu_\nu (r) \equiv r^4 \left( K^\mu_\nu - \kappa^\mu_\nu + 3 \gamma^\mu_\nu - \frac{1}{2} G^\mu_\nu \right),$$

where $G^\mu_\nu$ is associated with $\gamma_{\mu \nu}$. The last two terms are counter-terms which remove divergences near the boundary $r = \infty$. 
\[ \tilde{T}_0^0 = -3(1 - 4\epsilon b_1) + \frac{\epsilon}{2r} \left\{ -6rk + 4r^4\partial\beta - 4\partial j - r^3\partial_i\partial_j\alpha_{ij} + 18(r^5 - r)h \right. \\
\left. + 6(r^6 - r^2)\partial_r h + 2r^3\partial^2 h + 6r^4\partial_v h \right\}, \]

\[ \tilde{T}_i^0 = \frac{\epsilon}{2r^4} \left\{ 2 \left[ 4r^4\beta_i - 4(r^4 - 1)j_i + r^7\partial_v\beta_i - r^3\partial_i k + (r^5 - r)\partial_r j_i \right] \\
- r^2 \left[ -\partial^2 j_i + \partial_i\partial_j + r^4\partial_v\partial_k\alpha_{ik} - 2r^4\partial_v\partial_i h - 3r^5\partial_i h \right] \right\}, \]

\[ \tilde{T}_0^i = -\frac{\epsilon}{2r^3} \left\{ 2 \left[ 4r^3\beta_i - 4r^3j_i + r^6\partial_v\beta_i - r^2\partial_i k + (r^4 - 1)\partial_r j_i \right] \\
+ r \left[ \partial^2 j_i - \partial_i\partial_j - r^4\partial_v\partial_k\alpha_{ik} - 2r^4\partial_v\partial_i h - 3(r^6 - r^2)\partial_i h \right] \right\}, \]

\[ \tilde{T}_i^j = \delta_i^j(1 - 4\epsilon b_1) + \frac{\epsilon}{2r^4} \delta_i^j \left\{ r^2 \left[ -\partial^2 k + (1 - r^4)\partial_k\partial_l\alpha_{kl} + 2\partial_v\partial j \right] \\
- 2 \left[ (1 - r^4)k - 2r^7\partial\beta + 2r^3\partial j - r^3\partial_v k + (r^5 - r)\partial_r k \right] + r^6\partial^2 h \\
- 2r^6\partial^2 h + 2 \left[ (3 - 12r^4 + 9r^5) h + (r^3 - r^7)\partial_v h + (2r - 4r^5 + 2r^9)\partial_r h \right] \right\} \\
+ \frac{\epsilon}{2r^2} \left\{ -2r \left[ 2r^4\partial_{(i}\beta_{j)} - 2\partial_{(i\partial_j)} + r^4\partial_v\alpha_{ij} + (r^6 - r^2)\partial_r\alpha_{ij} \right] - r^4\partial_i\partial_j h \\
+ \left[ \partial_i\partial_j k + (1 - r^4)\partial^2\alpha_{ij} + 2(r^4 - 1)\partial_k\partial_{(i\alpha_{j)}k} - 2\partial_v\partial_{(i\partial_j)} + r^4\partial^2_v\alpha_{ij} \right] \right\}, \]
Approaching the boundary

\[ j_i \to -i \omega r^3 \beta_i - \frac{1}{3} r^2 \partial_i \partial \beta + O \left( \frac{1}{r} \right), \]

\[ \alpha_{ij} \to \left( \frac{2}{r} - \frac{\eta(\omega, q^2)}{4r^4} \right) \sigma_{ij} - \frac{\zeta(\omega, q^2)}{4r^4} \pi_{ij} + O \left( \frac{1}{r^5} \right). \]

\[ k \to \frac{2}{3} \left( r^3 + i \omega r^2 \right) \partial \beta + O \left( \frac{1}{r^2} \right), \quad \text{as} \quad r \to \infty \]

\[ h = 0 \]

The dissipative part of the stress tensor

\[ \Pi_{ij} = - \left[ \eta(\omega, q^2) \sigma_{ij} + \zeta(\omega, q^2) \pi_{ij} \right] \]
Einstein equations for the metric corrections

Dynamical equations:

\[ \mathbf{E}_{rr} = 0 : \quad 5 \partial_r h + r \partial_r^2 h = 0. \]

\[ \mathbf{E}_{rv} = 0 : \quad 3 r^2 \partial_r k = 6 r^4 \partial \beta + r^3 \partial_v \partial \beta - 2 \partial j - r \partial_r \partial_j - r^3 \partial_i \partial_j \alpha_{ij}. \]

\[ \mathbf{E}_{ri} = 0 : \quad -\partial_r^2 j_i = (\partial^2 \beta_i - \partial_i \partial \beta) + 3r \partial_v \beta_i - \frac{3}{r} \partial_r j_i + r^2 \partial_r \partial_j \alpha_{ij}. \]

\[ \mathbf{E}_{ij} = 0 : \quad (r^7 - r^3) \partial_r^2 \alpha_{ij} + (5r^6 - r^2) \partial_r \alpha_{ij} + 2r^5 \partial_v \partial_r \alpha_{ij} + 3r^4 \partial_v \alpha_{ij} \]

\[
+ r^3 \left\{ \partial^2 \alpha_{ij} - \left( \partial_i \partial_k \alpha_{jk} + \partial_j \partial_k \alpha_{ik} - \frac{2}{3} \delta_{ij} \partial_k \partial_l \alpha_{kl} \right) \right\}
+ \left( \partial_i j_j + \partial_j j_i - \frac{2}{3} \delta_{ij} \partial_j \right) - r \partial_r \left( \partial_i j_j + \partial_j j_i - \frac{2}{3} \delta_{ij} \partial_j \right)
+ 3r^4 \left( \partial_i \beta_j + \partial_j \beta_i - \frac{2}{3} \delta_{ij} \partial \beta \right) + r^3 \partial_v \left( \partial_i \beta_j + \partial_j \beta_i - \frac{2}{3} \delta_{ij} \partial \beta \right) = 0. \]
Holographic RG flow-type equations

\( j_i \) and \( \alpha_{ij} \) are linear functionals of \( \beta_i \). They can be uniquely decomposed as

\[
j_i = a(\omega, q, r) \beta_i + b(\omega, q, r) \partial_i \partial \beta
\]

\[
\alpha_{ij} = 2c(\omega, q, r) \sigma_{ij} + d(\omega, q, r) \pi_{ij},
\]

The Einstein equations reduce to ordinary diff equations

\[
\begin{align*}
 r \partial_r^2 a - 3 \partial_r a - q^2 r^3 \partial_r c - 3i \omega r^2 - q^2 r &= 0 \\
 r \partial_r^2 b - 3 \partial_r b + \frac{1}{3} r^3 \partial_r c - \frac{2}{3} r^3 q^2 \partial_r d - r &= 0 \\
 (r^7 - r^3) \partial_r^2 c + (5r^6 - r^2) \partial_r c - 2i \omega r^5 \partial_r c - r \partial_r a + a - 3i \omega r^4 c + 3r^4 - i \omega r^3 &= 0 \\
 (r^7 - r^3) \partial_r^2 d + (5r^6 - r^2) \partial_r d - 2i \omega r^5 \partial_r d + \frac{q^2}{3} r^3 d - 3i \omega r^4 d + 2b - 2r \partial_r b - \frac{2}{3} r^3 c &= 0.
\end{align*}
\]
Using dynamical Einstein equations, we have constructed an "off-shell" $T^{\mu\nu}$

Constraint equations

\[ E_{vv} = 0 \quad \text{and} \quad E_{vi} = 0 \]

are equivalent to the stress tensor conservation law

\[ \partial_\mu T^{\mu\nu} = 0 \]

which determines the temperature and velocity profiles as functions of time, provided initial configuration is specified.
Conclusions

- We have found that all order dissipative terms of a weakly perturbed conformal fluid are fully accounted for by two viscosity functions $\eta(\omega, q^2)$ and $\zeta(\omega, q^2)$.

- We have derived a closed form linear holographic RG flow-type equations for the viscosity functions.

- At large momenta, the effective viscosity is a decreasing function of both frequency and momentum, the behavior consistent with causality restoration.

- arXiv:1502....: For a weakly curved background space

$$\Pi_{\mu\nu} = -2\eta \nabla_\mu u_\nu - \zeta \nabla_\mu \nabla_\nu u + \kappa u^\alpha u^\beta C_{\mu\alpha\nu\beta} + \rho u^\alpha \nabla^\beta C_{\mu\alpha\nu\beta} + \xi \nabla^\alpha \nabla^\beta C_{\mu\alpha\nu\beta} - \theta u^\alpha \nabla_\alpha R_{\mu\nu},$$

$\kappa(\omega, q^2)$, $\rho(\omega, q^2)$, $\xi(\omega, q^2)$, and $\theta(\omega, q^2)$ are gravitational susceptibilities.

- arXiv:1503....: boundary hydrodynamics dual to Einstein-Gauss-Bonnet gravity

$$R - 2\Lambda \rightarrow R - 2\Lambda + \alpha_{GB} L_{GB}, \quad L_{GB} = R_{MNPQ} R^{MNPQ} - 4R_{MN} R^{MN} + R^2$$

Viscosity $\eta(\omega, q^2, \alpha_{GB})$. We derive causality induced constraints on $\alpha_{GB}$.