Convergent series for QCD

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Motivation

- We want to have convergent series for QCD!
- Standard perturbation theory does not converge and its series are asymptotic
- The reason is in the illegal interchange of $\sum$ and $\int$
- Is it possible to improve the perturbation expansions?

Introduction

- For the alternating series and ordinary integrals the legality of the interchange of $\sum$ and $\int$ is regulated by Fubini's theorem (It doesn't exist for the path integrals!)
- In case of positive series one always can change the order of $\sum$ and $\int$.

Two approaches

- It is possible to regularize initial integrals to satisfy Fubini's theorem (Works mostly for lattice integrals).

  Belokurov, Solov'ov, Shavgulidze 1996; Meurice 2002

- One can change the initial approximation to make the series positive (or to satisfy Fubini's theorem). This gives an universal approach to scalar field theories.

  Ushveridze 1983, 1984

Importance of the initial approximation

Consider an integral

$$I(g) = \int_{-\infty}^{+\infty} dx \, e^{-x^2 - gx^4}$$

and different ways to construct perturbation series:

1)

$$I(g) = \int_{-\infty}^{+\infty} dx \, e^{-x^2 - gx^4} \quad = $$

$$\int_{-\infty}^{+\infty} dx \sum_{n} \frac{(-x^2 - gx^4)^n}{n!} \quad = \quad \infty - \infty + \infty - \ldots$$

2)

$$I(g) = \int_{-\infty}^{+\infty} dx\, e^{-x^2} e^{-gx^4} = \int_{-\infty}^{+\infty} dx\, e^{-x^2} \sum_n \frac{(-gx^4)^n}{n!}$$

$$= \sqrt{\pi} - \frac{3}{4}\sqrt{\pi} g + \frac{105}{32}\sqrt{\pi} g^2 - \frac{3465}{128}\sqrt{\pi} g^3 + \ldots$$

3)

$$I(g) = \int_{-\infty}^{+\infty} dx\, e^{-gx^4} e^{-x^2} = \int_{-\infty}^{+\infty} dx\, e^{-gx^4} \sum_n \frac{(-x^2)^n}{n!}$$

$$= \sum_n \frac{\left((-1)^{2n} + 1\right) g^{-\frac{n}{2}-\frac{1}{4}} \Gamma\left(\frac{n}{2} + \frac{1}{4}\right)}{4\Gamma(n+1)}$$

The quality of the expansion depends on the initial approximation!!!

Non-analytical functions

The Taylor series of the function

$$e^{-\frac{1}{g}},$$

when $g \to 0$ is

$$0 + 0 + 0 + 0 + ...$$

Assume $g \to c$, where $c > 0$, then

$$e^{-\frac{1}{g}} = e^{-1/c} + \frac{e^{-1/c}(g-c)}{c^2} - \frac{(2c-1)e^{-1/c}(g-c)^2}{2c^4} + \frac{\left(6c^2-6c+1\right)e^{-1/c}(g-c)^3}{6c^6} + ...$$

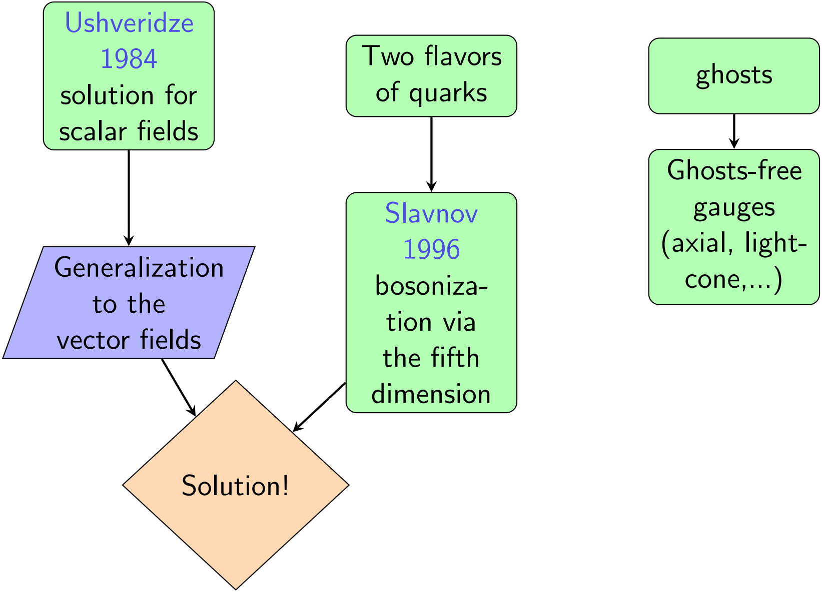
Statement of the problem

- There is a solution for an arbitrary scalar field theory (Ushveridze 1984)
- Is it possible to generalize it to QCD?

What to do with

- gauge fields,
- quarks,
- ghosts?

Scheme of the solution



Main idea of the Ushveridze method

Consider a normalized partition function of the scalar field

$$Z = \frac{1}{Z_0} \int D\phi \, e^{-S[\phi]} \,, \quad Z_0 = \int D\phi \, e^{-S_0[\phi]}$$

Existence of the path integral $\Longrightarrow$ $Z < \infty$

$$S[\phi] = N[\phi] + (S[\phi] - N[\phi])$$

$$Z = \frac{1}{Z_0} \int D\phi \, e^{-N[\phi]} \sum_n \frac{(N[\phi] - S[\phi])^n}{n!} \,.$$

When $(N[\phi] - S[\phi]) \geq 0$,

$$Z = \frac{1}{Z_0} \sum_n \frac{1}{n!} \int D\phi \, e^{-N[\phi]} (N[\phi] - S[\phi])^n$$

The convergence of the series is guaranteed by the condition $Z < \infty$.

Choice of the non-perturbed action $N[\phi]$

Q - is a maximal power of the field ϕ in action $S[\phi]$
We define

$$N[\phi] = \sum_{q=2}^{Q} \sigma_q \|\phi\|^q \,, \quad \sigma_q > 0$$

$$\|\phi\| = \Big[\int d^D x \, \phi(x) \Big(\sum_{u=0}^{U} \omega_u (-\partial_\nu^2)^u \Big) \phi(x) \Big]^{1/2} , \quad \omega_u > 0$$

According to the Hölder inequality and Sobolev theorem, there are such $\sigma_q > 0$ and U, that

$$N[\phi] \geq S[\phi]$$

Calculation of the correlation functions

Correlation function is given by

$$G(K,S) = \int D\phi\, K[\phi]\, \exp\{-S[\phi]\}$$

$$K[\phi] = \prod_{i=1}^{k} \phi(x_i)$$

We split the action as

$$S[\phi] = N[\phi] + (S[\phi] - N[\phi])\,, \quad N[\phi] = \sum_{q=2}^{Q} \sigma_q \|\phi\|^q$$

$$G(K,S) = \sum_{n=0}^{\infty} G_n(K,S)$$

$$G_n(K,S) = \frac{1}{n!} \int D\phi\, K[\phi]\, \{N[\phi] - S[\phi]\}^n \exp\{-N[\phi]\}$$

Calculation of the correlation functions

$$G_n(K,S) = \frac{1}{n!}\int D\phi\, K[\phi]\,\{\sum_{q=2}^{Q}\sigma_q\|\phi\|^q - S[\phi]\}^n \exp\{-\sum_{q=2}^{Q}\sigma_q\|\phi\|^q\}$$

$$= \frac{1}{n!}\int_0^\infty dt\, \exp\Big(-\sum_q \sigma_q t^q\Big)\int D\phi\, K[\phi]\,\delta(t-\|\phi\|)\cdot \Big(\sum_q \sigma_q t^q - S[\phi]\Big)^n$$

Dimensional regularization $\Longrightarrow D(c\phi) = D\phi$.

$$G_n(K,S) = \frac{1}{n!}\int_0^\infty dt\, t^{k-1}\exp\Big(-\sum_q \sigma_q t^q\Big)\cdot \int D\phi\, K[\phi]\,\delta(1-\|\phi\|)\Big(\sum_q \sigma_q t^q - S[t\phi]\Big)^n$$

Typical term of the series

$$\text{Integral}(t) \cdot \int D\phi\, K[\phi] e^{-\|\phi\|^2} \prod_q (-S_q[\phi]/\sigma_q)^{m_q} \,,$$

$S_q[\phi]$ - part of the action containing vertices with q legs

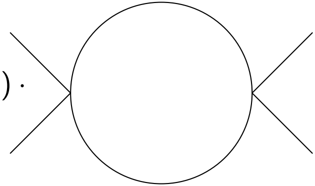
We have Feynman diagrams with new propagator

$$f(p) = \left(\sum_{l=0}^{L} \omega_l p^{2l} \right)^{-1}, \quad \omega_l > 0$$

Small $L \implies$ UV divergences in diagrams $\implies$ one can use RG (Honkonen,Nalimov 1999).

Sufficiently large $L \implies$ there are no UV divergences in diagrams!

Diagram

$Integral(t) \cdot$ 

$= f(p)$

Bosonization of QCD

Integration over fermion fields leads to the determinant

$$\det(D+m)^2 = \det(\gamma_5(D+m)\gamma_5(D+m)) = \det(-D^2+m^2) \equiv \det(B^2+m^2)\,,$$

Slavnov 1996, 1997:

$$\begin{aligned}
\det(B^2+m^2) &= \lim_{L\to\infty}\int D\phi^*_{i,\alpha} D\phi_{i,\alpha} D\xi^*_{i,\alpha} D\xi_{i,\alpha}\ \exp\Bigg\{ -\int_0^L ds \int dx \\
&\quad \Big[(\partial_s \phi^*_{i,\alpha}(x,s) - iB_{ij,\alpha\beta}(x)\phi^*_{j,\beta}(x,s)) \\
&\cdot \quad (\partial_s \phi_{i,\alpha}(x,s) + iB_{ij,\alpha\beta}(x)\phi_{j,\beta}(x,s)) \\
&+ \quad \sqrt{L}(\xi^*_{i,\alpha}(x)(m\delta_{ij}\delta_{\alpha\beta} + iB_{ij,\alpha\beta}(x))\phi_{j,\beta}(x,s) + \text{h.c.}) \\
&- \quad \frac{1}{2m}\xi^*_{i,\alpha}(x)\xi_{i,\alpha}(x)\Big]\Bigg\}
\end{aligned}$$

Conclusions

- We have constructed the convergent series for QCD without infrared and ultraviolet divergences.
- It gives a new powerful tool for the analytical studies... and PROBABLY for the numerical computations.

Conclusions

- The generalization to other gauge groups is evident. All steps used to derive the convergent series for QCD are also valid for the pure Yang-Mills theories.
- There is a chance to derive the convergent series for QCD at finite chemical potential (Non-hermitian bosonization, Sazonov 2014)

Conclusions

- The convergent series does not rely on the expansion in small g and, therefore, automatically takes into account such non-analytic contributions.
- CS has all advantages of non-perturbative approaches.
- The relation of the CS to the Feynman diagrammatic technique should provide the possibility of the non-perturbative generalizations of conjectures, proved early only within the framework of the standard perturbation theory. (Large N_c QCD at finite chemical potential $\longleftrightarrow$ Large N_c QCD at finite isospin chemical potential)
- All details - POSTER SESSION!

Thank you for your attention!