Bachelor Thesis (BSc)

SEARCH FOR GRIBOV COPIES USING A MODIFIED INSTANTON CONFIGURATION

by

Michael Mandl

Supervisor: Axel Maas

2017

Abstract

In non-Abelian gauge theories of the Yang-Mills type, fixing a gauge is inevitable in order to perform calculations. However, according to the Gribov-Singer ambiguity, gauge fixing beyond perturbation theory is impossible by only imposing local conditions. In the Gribov-Zwanziger scenario, this is solved by restricting to field configurations that lie within the first Gribov region, where the Faddeev-Popov operator is positive definite. Alternatively, it should also be possible to average over all Gribov copies in a certain way in order to deal with the Gribov-Singer ambiguity. The aim of this thesis is to find Gribov copies outside the first Gribov region - and consequently normalizable eigenstates corresponding to negative eigenvalues of the Faddeev-Popov operator - in $SU(2)$ Yang-Mills theory.

To this end, the instanton field configuration is modified in certain different ways: first, a constant modification is investigated and the eigenstates determined algebraically. Next, a number of different modifications are analyzed by using power series expansions, the termination conditions of which are looked for as well. Finally, some general considerations are made on modifications of the instanton configuration. Despite the results not meeting all of the restrictions imposed on them, some light is shed on how Gribov copies outside the first Gribov region might look like.

Contents

1 Introduction **3**
- 1.1 Gauge theories . . . 3
- 1.2 Yang-Mills theory . . . 3
- 1.3 Gauge fixing . . . 4
- 1.4 Gribov regions . . . 5

2 Modification of the instanton configuration **6**
- 2.1 The instanton field configuration . . . 6
- 2.2 Modifying the instanton ansatz . . . 6
- 2.3 Restriction on u(r) . . . 8
- 2.4 Lagrangian and action . . . 8

3 Constant u(r) **9**
- 3.1 Eigenfunctions . . . 9
- 3.2 Normalization . . . 10
- 3.3 Lagrangian and action . . . 11

4 Power series expansions **12**
- 4.1 Exponential u(r) . . . 12
- 4.2 A more simple u(r) . . . 15
- 4.3 Half-integer exponents . . . 16
- 4.4 Second-order polynomial as denominator of u(r) . . . 18
- 4.5 A general approach . . . 19
- 4.6 A general rational approach . . . 20

5 Summary and outlook **22**

1 Introduction

1.1 Gauge theories

To date, the standard model of particle physics is the most successful theory in theoretically describing particle physics, containing three of the four fundamental interactions: the strong and weak interactions and electromagnetism. Additionally, it covers the Higgs-mechanism, but does not describe gravitational interactions. Since all of the former are so-called gauge theories, it has become one of the most important tasks of modern physics to study gauge theories and understanding them has become an inevitability in the field of particle physics.

The most prominent feature of gauge theories is that the physically observable quantities are invariant under certain types of transformations of the fields, called gauge transformations. The best known example of a gauge transformation is to be found in electromagnetism, where the transformations of the vector potential $\boldsymbol{A}$ and the scalar potential V

$$\boldsymbol{A} \to \boldsymbol{A}' = \boldsymbol{A} + \boldsymbol{\nabla}\chi$$

$$V \to V' = V - \frac{\partial \chi}{\partial t},$$

(with t denoting time and χ being an arbitrary differentiable function depending on space and time) leave the physically observable electric and magnetic fields $\boldsymbol{E}$ and $\boldsymbol{B}$

$$\boldsymbol{E} = -\boldsymbol{\nabla}V - \frac{\partial \boldsymbol{A}}{\partial t}$$

$$\boldsymbol{B} = \boldsymbol{\nabla} \times \boldsymbol{A}$$

unchanged.

1.2 Yang-Mills theory

The most important gauge theories are those of the Yang-Mills type, as they are used for describing both the strong and the electroweak interaction, the latter being a unification of the weak interaction and electromagnetism. Thus, this thesis will restrict to Yang-Mills-type gauge theories exclusively. Electromagnetism - arguably the most famous gauge theory - is an example of an Abelian Yang-Mills theory. However, the main focus of this thesis shall be on non-Abelian Yang-Mills theories, the mathematical formalism of which is much more complex.

In general, every gauge theory has an underlying symmetry group of gauge transformations. In the case of Yang-Mills theories, possible underlying groups are for instance the special unitary groups $SU(n)$, the Lie groups of unitary $n \times n$-matrices with determinant 1. From here onwards, $SU(2)$ will be considered exclusively.

The central elements of interest in the following will be gauge fields, also called field configurations, which will be denoted by A_μ^a[1]. The Lagrangian of Yang- Mills theory is [1][2]

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu}^a F_{\mu\nu}^a \tag{1}$$

with

$$F_{\mu\nu}^a = \partial_\mu A_\nu^a - \partial_\nu A_\mu^a + g f^{abc} A_\mu^b A_\nu^c. \tag{2}$$

Herein, g denotes the coupling constant and f^{abc} are constants that vanish in Abelian Yang-Mills theories. In the case of $SU(2)$ as the underlying symmetry group, the numbers f^{abc} are given by the totally antisymmetric Levi-Civita tensor ϵ^{abc} [1]. The Lagrangian (1) is invariant under finite gauge transformations of the form [2]

$$\begin{aligned} A_\mu \to A_\mu^{(h)} &= h A_\mu h^{-1} + h \partial_\mu h^{-1} \\ A_\mu &= \tau^a A_\mu^a \\ h &= \tau^a \phi^a, \end{aligned}$$

with arbitrary functions ϕ^a and with τ^a denoting for $SU(2)$ the Pauli matrices

$$\tau^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \tau^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad \tau^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

1.3 Gauge fixing

As seen in the previous subsection, there is a set of field configurations $A_\mu^{(h)}$ connected by gauge transformations h, for which the Lagrangian (1) is the same. This set is called the gauge orbit $\mathcal{G}(A_\mu)$ of the field configuration [2]:

$$\mathcal{G}(A_\mu) = \{A_\mu^{(h)} \ \forall h\}.$$

Even though the Lagrangian is gauge-invariant, there are still important quantities that do depend on the choice of gauge, making it necessary to fix a gauge for all calculations. One could - for example - require the field configurations to obey the Landau gauge condition

$$\partial_\mu A_\mu^a = 0, \tag{3}$$

which would actually fix the gauge in perturbation theory. Beyond perturbation theory, however, (3) does not lead to a unique solution anymore [3]. This phenomenon is called the Gribov-Singer ambiguity and the different possible solutions are called Gribov copies. The remaining part of the gauge orbit fulfilling the gauge condition is called the residual gauge orbit [2]. A necessary requirement in order to fully fix the gauge is to be able to somehow pick one single Gribov copy out of the residual gauge orbit, which was, however, shown to be impossible to achieve by local gauge conditions (like (3)) alone [4].

[1] In this thesis, everything is given in four-dimensional Euclidean space. As for the indices, Greek letters denote space-time coordinates, while Roman letters denote coordinates in color space.

[2] Note that the Einstein summation convention and natural units $\hbar = c = 1$ are used here and in the following.

1.4 Gribov regions

In the Gribov-Zwanziger scenario, it is assumed that the Gribov-Singer ambiguity could be solved by restricting to field configurations for which the Faddeev-Popov operator

$$M^{ab} = -\partial_\mu \left(\partial_\mu \delta^{ab} + g f^{abc} A^c_\mu \right) \tag{4}$$

is positive definite [3,5]. The region in configuration space for which this restriction holds is called the first Gribov region, a convex region with a boundary termed the first Gribov horizon. At the first Gribov horizon, a zero eigenvalue appears, becoming negative beyond the horizon. There are more Gribov regions outside of the first one, being separated by additional Gribov horizons and supporting more and more negative eigenvalues [3]. As to date most research has been done on the first Gribov region, the aim of this thesis shall be to focus on regions outside the first Gribov region in search of normalizable eigenstates that host negative eigenvalues. This is important, as it is possible theoretically to average over all Gribov copies in every Gribov region in order to solve the Gribov-Singer ambiguity [2].

2 Modification of the instanton configuration

2.1 The instanton field configuration

The eigenequation of the Faddeev-Popov operator (4)

$$M^{ab}\phi^b = \omega^2\phi^a \tag{5}$$

leads to zero modes for an instanton field configuration characterized by the gauge fields

$$A^a_\mu = \frac{1}{g}\frac{2}{r^2+\lambda^2}r_\nu\zeta^a_{\nu\mu} \tag{6}$$

$$r_\mu = (x,y,z,t)^T\,,$$

wherein $\zeta^a_{\mu\nu}$ denote the t'Hooft symbols

$$\zeta^1_{\mu\nu} = \begin{pmatrix} 0 & 0 & 0 & 1\\ 0 & 0 & 1 & 0\\ 0 & -1 & 0 & 0\\ -1 & 0 & 0 & 0\end{pmatrix},\ \zeta^2_{\mu\nu} = \begin{pmatrix} 0 & 0 & -1 & 0\\ 0 & 0 & 0 & 1\\ 1 & 0 & 0 & 0\\ 0 & -1 & 0 & 0\end{pmatrix},\ \zeta^3_{\mu\nu} = \begin{pmatrix} 0 & 1 & 0 & 0\\ -1 & 0 & 0 & 0\\ 0 & 0 & 0 & 1\\ 0 & 0 & -1 & 0\end{pmatrix}.$$

Because the Landau gauge condition (3) holds for (6), the instanton field configuration is admissible in Landau-gauge [6].

2.2 Modifying the instanton ansatz

In order to find normalizable eigenstates sustaining negative eigenvalues w^2, we modify (6) by replacing the term $\frac{2}{r^2+\lambda^2}$ with any function $-u(r)$ depending only on the radial distance[3]:

$$A^a_\mu = -\frac{u(r)}{g}r_\nu\zeta^a_{\nu\mu} \tag{7}$$

$$r = \sqrt{x^2+y^2+z^2+t^2}.$$

This modification is the only significant difference to the calculations done in the original work [6]; the remainder of subsection 2.2 follows [6] in close analogy.

Inserting (7) into the Faddeev-Popov operator (4), the eigenequation (5) reads

$$-\partial^2\phi^a + u(r)f^{abc}r_\nu\zeta^c_{\nu\mu}\partial_\mu\phi^b = \omega^2\phi^a$$

and by exploiting the antisymmetry of f^{abc}

$$\partial^2\phi^a + u(r)f^{abc}r_\nu\zeta^b_{\nu\mu}\partial_\mu\phi^c = -\omega^2\phi^a. \tag{8}$$

We identify the terms $r_\nu\zeta^b_{\nu\mu}\partial_\mu$ as angular momentum operators

$$L_{\alpha\beta} = i(x_\alpha\partial_\beta - x_\beta\partial_\alpha)$$

[3] In this thesis, at some points hyperspherical coordinates $r_\mu = (x,y,z,t)^T = (r\cos\phi\sin\theta\sin\eta, r\sin\phi\sin\theta\sin\eta, r\cos\theta\sin\eta, r\cos\eta)^T$ are used.

and rewrite eq.(8) as the three equations

$$\left(\partial^2+\omega^2\right)\phi^1+\frac{u(r)}{i}\left(\left(L_{34}+L_{12}\right)\phi^2+\left(-L_{24}+L_{13}\right)\phi^3\right)=0 \tag{9a}$$

$$\left(\partial^2+\omega^2\right)\phi^2+\frac{u(r)}{i}\left(-\left(L_{34}+L_{12}\right)\phi^1+\left(L_{14}+L_{23}\right)\phi^3\right)=0 \tag{9b}$$

$$\left(\partial^2+\omega^2\right)\phi^3+\frac{u(r)}{i}\left(\left(L_{24}-L_{13}\right)\phi^1-\left(L_{14}+L_{23}\right)\phi^2\right)=0. \tag{9c}$$

Further simplification is possible by introducing the set of operators

$$L^1=\frac{1}{2}\left(L_{14}+L_{23}\right),\ L^2=\frac{1}{2}\left(L_{13}-L_{24}\right),\ L^3=\frac{1}{2}\left(L_{34}+L_{12}\right)$$

with

$$\boldsymbol{L}^2=\left(L^1\right)^2+\left(L^2\right)^2+\left(L^3\right)^2$$

and by using that [6]

$$\partial^2=\frac{1}{r^3}\partial_r r^3\partial_r-\frac{4\boldsymbol{L}^2}{r^2}.$$

As shown in [6], $\mathbf{L}^2$ can be replaced by its eigenvalues $l(l+1)$. Eqs.(9) now read

$$\left(\frac{1}{r^3}\partial_r r^3\partial_r-\frac{4l(l+1)}{r^2}+\omega^2\right)\phi^1+\frac{2u(r)}{i}\left(L^3\phi^2+L^2\phi^3\right)=0 \tag{10a}$$

$$\left(\frac{1}{r^3}\partial_r r^3\partial_r-\frac{4l(l+1)}{r^2}+\omega^2\right)\phi^2+\frac{2u(r)}{i}\left(-L^3\phi^1+L^1\phi^3\right)=0 \tag{10b}$$

$$\left(\frac{1}{r^3}\partial_r r^3\partial_r-\frac{4l(l+1)}{r^2}+\omega^2\right)\phi^3+\frac{2u(r)}{i}\left(-L^2\phi^1-L^1\phi^2\right)=0. \tag{10c}$$

We divide both sides by $\frac{u(r)}{2}$, separating eqs.(10) each into a radial and an angular part. For the former, we introduce the operator

$$D_r=\frac{2}{u(r)}\left(\frac{1}{r^3}\partial_r r^3\partial_r-\frac{4l(l+1)}{r^2}+\omega^2\right)$$

and for the latter the unitary matrix

$$L_I=\frac{4}{i}\begin{pmatrix}0 & L^3 & L^2\\ -L^3 & 0 & L^1\\ -L^2 & -L^1 & 0\end{pmatrix}.$$

Eqs.(10) now take the form of the matrix equation

$$\left(\mathbb{1}D_r+L_I\right)\boldsymbol{\phi}=0. \tag{11}$$

If $\{\boldsymbol{l}_i\}$ is the eigenbasis of L_I, we can expand the solutions of eq.(11) in this basis: $\boldsymbol{\phi}=\sum\phi_{li}\boldsymbol{l}_i$. The fully decoupled radial equation now reads

$$\left(D_r+c_i\right)\phi_{li}=0$$

with the eigenvalues c_i to the corresponding eigenvectors $\boldsymbol{l_i}$ of L_I. There are only three different eigenvalues possible: $c_i=4,\ -4l$ and $4(l+1)$ with multiplicities $2l+1$, $2l+3$ and $2l-1$ respectively [6]. Since ϕ_{li} does not depend on i, we can replace it by ϕ_{lc} for the following. The radial equation now takes the form

$$\partial_r^2\phi_{lc}+\frac{3}{r}\partial_r\phi_{lc}+\left(\omega^2-\frac{4l(l+1)}{r^2}+\frac{cu(r)}{2}\right)\phi_{lc}=0. \tag{12}$$

2.3 Restriction on u(r)

In order for the modified field configuration to still be in Landau gauge, i.e. for the Landau gauge condition (3) to hold for (7), there is a restriction we need to impose on the modification $u(r)$.

$$\begin{aligned}\partial_\mu A^a_\mu &= -\frac{1}{g}\left(r_\nu \zeta^a_{\nu\mu}\frac{\partial u(r)}{\partial r_\mu} + u(r)\frac{\partial r_\nu}{\partial r_\mu}\zeta^a_{\nu\mu}\right)\\ &= -\frac{1}{g}\left(r_\nu \zeta^a_{\nu\mu}\frac{\partial u(r)}{\partial r_\mu} + u(r)\delta_{\nu\mu}\zeta^a_{\nu\mu}\right)\end{aligned}$$

with $\delta_{\nu\mu}$ denoting the Kronecker-delta. The second term is a sum over the diagonal terms of the antisymmetric t'Hooft-symbols $\zeta^a_{\nu\mu}$, which equals to zero. Furthermore,

$$\frac{\partial u(r)}{\partial r_\mu} = \frac{\partial u(r)}{\partial r}\frac{\partial r}{\partial r_\mu} = \frac{\partial u(r)}{\partial r}\frac{r_\mu}{r}$$

and hence

$$\partial_\mu A^a_\mu = -\frac{1}{g}\frac{\partial u(r)}{r\partial r} r_\nu \zeta^a_{\nu\mu} r_\mu. \tag{13}$$

Due to the structure of the t'Hooft symbols, the sum $r_\nu \zeta^a_{\nu\mu} r_\mu$ vanishes necessarily. Thus, in order for (13) to equal zero and consequently for (7) to be in Landau gauge, a sufficient restriction is to require for $\frac{\partial u(r)}{r\partial r}$ to be finite for each r. However, configurations with $\frac{\partial u(r)}{r\partial r}$ diverging slower than r^{-2} as $r \to 0$ should also be admissible.

2.4 Lagrangian and action

We now look for a general expression of the Lagrangian (1) of the modified instanton field configuration (7). Straightforward calculation yields with $u'(r) = \frac{\partial u(r)}{\partial r}$:

$$\mathcal{L} = -\frac{3}{2g^2}\left(8u(r)^2 - 4r^2u(r)^3 + r^4u(r)^4 + 4ru(r)u'(r) + r^2u'(r)^2\right). \tag{14}$$

The action of a field configuration is an integral of (14) over space-time and should classically be finite.

3 Constant u(r)

We make a first simple ansatz, assuming $u(r) = u = const$. Eq.(12) now reads

$$\partial_r^2 \phi_{lc} + \frac{3}{r}\partial_r \phi_{lc} + k\phi_{lc} - \frac{4l(l+1)}{r^2}\phi_{lc} = 0 \tag{15}$$

$$k = \omega^2 + \frac{cu}{2}.$$

3.1 Eigenfunctions

After multiplying both sides of eq.(15) by r^2, we have

$$r^2\partial_r^2\phi_{lc} + 3r\partial_r\phi_{lc} + r^2k\phi_{lc} - 4l(l+1)\phi_{lc} = 0. \tag{16}$$

We now set $\phi_{lc} = \frac{1}{r}\phi$, so that eq.(16) becomes

$$r^2\partial_r\left(-\frac{1}{r^2}\phi + \frac{1}{r}\partial_r\phi\right) + 3r\left(-\frac{1}{r^2}\phi + \frac{1}{r}\partial_r\phi\right) + rk\phi - \frac{4l(l+1)}{r}\phi$$

$$= r^2\left(\frac{2}{r^3}\phi - \frac{1}{r^2}\partial_r\phi - \frac{1}{r^2}\partial_r\phi + \frac{1}{r}\partial_r^2\phi\right) + 3r\left(-\frac{1}{r^2}\phi + \frac{1}{r}\partial_r\phi\right) + rk\phi - \frac{4l(l+1)}{r}\phi$$

$$= r\partial_r^2\phi + \partial_r\phi - \frac{1}{r}\phi + rk\phi - \frac{4l(l+1)}{r}\phi = 0$$

$$r^2\partial_r^2\phi + r\partial_r\phi + \left(r^2k - (4l^2+4l+1)\right)\phi = 0$$

$$\left(r\sqrt{k}\right)^2\frac{\partial^2\phi}{\partial\left(r\sqrt{k}\right)^2} + \left(r\sqrt{k}\right)\frac{\partial\phi}{\partial\left(r\sqrt{k}\right)} + \left(\left(r\sqrt{k}\right)^2 - (2l+1)^2\right)\phi = 0.$$

This is an equation of the form

$$x^2\frac{d^2f}{dx^2} + x\frac{df}{dx} + \left(x^2 - \nu^2\right)f = 0,$$

a Bessel differential equation which is solved by Bessel functions of the first and second kind $J_\nu(x)$ and $Y_\nu(x)$. The $Y_\nu(x)$ are omitted here, being singular at the origin [7]. Thus,

$$\phi = J_{2l+1}\left(r\sqrt{k}\right) = J_{2l+1}\left(r\sqrt{\omega^2 + \frac{cu}{2}}\right).$$

The solutions of eq.(12) for constant $u(r)$ now read

$$\phi_{lc} = \frac{1}{r}J_{2l+1}\left(r\sqrt{\omega^2 + \frac{cu}{2}}\right). \tag{17}$$

As we require the argument of J_{2l+1} to be greater than zero and because $\omega^2 < 0$, the range of possible u is restricted.

3.2 Normalization

We check whether or not the solutions (17) to eq.(16) are normalizable, i.e. whether or not the norm

$$N = \int_0^\infty r^3 \left|\phi_{lc}\right| dr = \int_0^\infty r^2 \left| J_{2l+1}\left(r\sqrt{\omega^2 + \frac{cu}{2}} \right) \right| dr \tag{18}$$

is finite. As we assume (18) to diverge, we instead investigate the more simple integral

$$N' = \int_0^\infty r J_{2l+1}\left(r\sqrt{\omega^2 + \frac{cu}{2}} \right) dr, \tag{19}$$

which - should it diverge - ensures a divergent norm as well. Substituting $r\sqrt{\omega^2 + \frac{cu}{2}} \to v$, (19) becomes

$$N' = \frac{1}{\omega^2 + \frac{cu}{2}} \int_0^\infty v J_{2l+1}(v) dv.$$

The integration of Bessel functions of the first kind takes the form [7]

$$\int_0^z t^\mu J_\nu(t) dt = \frac{z^\mu \Gamma\left(\frac{\nu+\mu+1}{2}\right)}{\Gamma\left(\frac{\nu-\mu+1}{2}\right)} \times \sum_{k=0}^\infty \frac{(\nu+2k+1)\Gamma\left(\frac{\nu-\mu+1}{2}+k\right)}{\Gamma\left(\frac{\nu+\mu+3}{2}+k\right)} J_{\nu+2k+1}(z)$$

if $Re(\mu+\nu+1) > 0$, which holds, since in (19) $\mu = 1$ and $\nu = 2l+1$. This leads to

$$\int_0^z v J_{2l+1}(v) dv = \frac{z\Gamma\left(\frac{2l+3}{2}\right)}{\Gamma\left(\frac{2l+1}{2}\right)} \times \sum_{k=0}^\infty \frac{2(l+k+1)\Gamma\left(\frac{2l+1}{2}+k\right)}{\Gamma\left(\frac{2l+5}{2}+k\right)} J_{2(l+k+1)}(z).$$

To obtain N', we need to let z go towards infinity. The long-distance behaviour of Bessel functions of the first kind is [7]

$$\lim_{z\to\infty} J_\nu(z) = \sqrt{\frac{2}{\pi z}} \cos\left(z - \frac{1}{2}\nu z - \frac{1}{4}\pi \right).$$

Thus, the integral (19) finally reads

$$\begin{aligned} N' = \lim_{z\to\infty} & \frac{\sqrt{2z}}{\sqrt{\pi}\left(\omega^2 + \frac{cu}{2}\right)} \frac{\Gamma\left(\frac{2l+3}{2}\right)}{\Gamma\left(\frac{2l+1}{2}\right)} \\ & \times \sum_{k=0}^\infty \frac{2(l+k+1)\Gamma\left(\frac{2l+1}{2}+k\right)}{\Gamma\left(\frac{2l+5}{2}+k\right)} \cos\left(z - (l+k+1)z\pi - \frac{1}{4}\pi \right), \end{aligned}$$

a function of z, the convergence behaviour of which is non-trivial to predict, as it contains oscillating terms. However, since there are necessarily points at which the cosine does not vanish and the first term is proportional to $\sqrt{z}$, it is assumed that (19) and consequently (18) do not converge. The solutions (17) are thus non-normalizable.

3.3 Lagrangian and action

Lastly, we seek to calculate the Lagrangian and subsequently the action of the modified instanton configuration (7) with a constant $u(r)$. Inserting u into (14) yields

$$\mathcal{L} = -\frac{3}{2g^2}\left(8u^2 - 4r^2u^3 + r^4u^4\right)$$

as the terms containing derivatives of u vanish. The action is essentially an integral of the Lagrangian over space-time. However, as the upper integral limit for the integration over r is infinity, the action does not converge, giving further evidence that the configuration (7) with a constant u is not a classically admissible field configuration.

4 Power series expansions

We now try to solve eq.(12) for selected $u(r)$ by power series expansions of the solutions ϕ_{lc}. Ideally, the $u(r)$ are chosen so that the field configurations (7) remain in Landau-gauge and thus (13) equals zero. Additionally, their action should be finite and therefore the integral of the Lagrangian (14) over space-time should not diverge. However, these two requirements do not hold for each of the $u(r)$ analyzed below.

4.1 Exponential u(r)

The first configuration to be investigated has a modification

$$u(r) = e^{-r}(1+r), \tag{20}$$

fulfilling both restrictions stated above. Eq.(12) now takes the form

$$\partial_r^2 \phi_{lc} + \frac{3}{r}\partial_r \phi_{lc} + \left(\omega^2 - \frac{4l(l+1)}{r^2} + \frac{c}{2}e^{-r}(r+1)\right)\phi_{lc} = 0. \tag{21}$$

We start off by investigating its asymptotic behaviour. For small r, eq.(21) becomes

$$\partial_r^2 \phi_{lc} + \frac{3}{r}\partial_r \phi_{lc} - \frac{4l(l+1)}{r^2}\phi_{lc} = 0$$

with the solutions $\phi_{lc,\ r\to 0} = C_1 r^{2l} + C_2 r^{-2(l+1)}$ [6], the second of which being singular at the origin, while for large r it looks like

$$\partial_r^2 \phi_{lc} + \frac{3}{r}\partial_r \phi_{lc} + \omega^2 \phi_{lc} = 0$$

and is solved by Bessel-functions $\phi_{lc,\ r\to\infty} = \frac{D_1}{r} J_n(\omega r) + \frac{D_2}{r} Y_n(\omega r)$ [6]. C_1, C_2, D_1 and D_2 are integration constants.

We now make the ansatz

$$\phi_{lc} = r^{2l} \sum_{i=0}^{\infty} a_i (-1)^i r^i \tag{22}$$

and evaluate the derivatives in eq.(21):

$$\frac{3}{r}\partial_r \phi_{lc} = 3r^{2l-2} \sum_{i=0}^{\infty} a_i (-1)^i r^i (2l+i)$$

$$\partial_r^2 \phi_{lc} = r^{2l-2} \sum_{i=0}^{\infty} a_i (-1)^i r^i (2l+i)(2l+i-1).$$

For the rest of the calculation we need to expand the exponential function in (20) into a power series as well. Using $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$, we have

$$\frac{c}{2}e^{-r}(1+r)\phi_{lc} = \frac{c}{2}r^{2l}\sum_{n=0}^{\infty}\frac{(-1)^n r^n}{n!}(1+r)\sum_{i=0}^{\infty}a_i(-1)^i r^i$$

$$= \frac{c}{2}r^{2l}\sum_{n=0}^{\infty}\frac{(-1)^n}{n!}\left(r^n + r^{n+1}\right)\sum_{i=0}^{\infty}a_i(-1)^i r^i$$

$$= \frac{c}{2}r^{2l}\sum_{i=0}^{\infty}a_i(-1)^i r^i + \frac{c}{2}r^{2l}\sum_{n=0}^{\infty}r^{n+1}\left(\frac{(-1)^n}{n!} - \frac{(-1)^n}{(n+1)!}\right)\sum_{i=0}^{\infty}a_i(-1)^i r^i$$

and finally

$$\frac{c}{2}e^{-r}(1+r)\phi_{lc} = \frac{c}{2}r^{2l}\sum_{i=0}^{\infty}a_i(-1)^i r^i + \frac{c}{2}r^{2l}\sum_{n=0}^{\infty}(-1)^n r^{n+1}\frac{n}{(n+1)!}\sum_{i=0}^{\infty}a_i(-1)^i r^i.$$

Hence, eq.(21) now takes the form

$$\sum_{i=0}^{\infty}a_i(-1)^i r^{2l}\Bigg[3r^{i-2}(2l+i) + r^{i-2}(2l+i)(2l+i-1) + \omega^2 r^i$$
$$-4l(l+1)r^{i-2} + \frac{c}{2}r^i + \frac{c}{2}\sum_{n=0}^{\infty}(-1)^n r^{n+i+1}\frac{n}{(n+1)!}\Bigg] = 0$$

$$\sum_{i=0}^{\infty}a_i(-1)^i r^{2l}\left[r^{i-2}i(4l+2+i) + r^i\left(\omega^2 + \frac{c}{2}\right) + \frac{c}{2}\sum_{n=0}^{\infty}(-1)^n r^{n+i+1}\frac{n}{(n+1)!}\right] = 0.$$

In order for this equation to hold, the coefficients of r^m have to vanish for each m individually. The coefficient of r^{2l+i-2} takes the form

$$a_i(-1)^i i(4l+i+2) + a_{i-2}(-1)^i\left(\omega^2 + \frac{c}{2}\right) + \frac{c}{2}\sum_{j+n=i-3}a_j(-1)^{j+n}\frac{n}{(n+1)!} = 0.$$

We can rewrite this to obtain the recurrence formula

$$a_i = \frac{-1}{i(4l+i+2)}\left(a_{i-2}\left(\omega^2 + \frac{c}{2}\right) + \frac{c}{2}\sum_{j+n=i-3}a_j(-1)^{i+j+n}\frac{n}{(n+1)!}\right)$$

$$a_i = \frac{-1}{i(4l+i+2)}\left(a_{i-2}\left(\omega^2 + \frac{c}{2}\right) - \frac{c}{2}\sum_{j+n=i-3}a_j\frac{n}{(n+1)!}\right), \tag{23}$$

which holds for $i \geq 1$[4], as $a_i = 0$ for $i < 0$. This leaves only a_0 to be determined. The coefficient of r^{2l-2} only contains a_0 and a vanishing factor, leading to

$$a_0 = D,$$

[4] Note that in the sum in (23) $n \geq 0$.

with D a normalization constant. Since every coefficient a_i in the recurrence formula contains a sum of previous coefficients, each iteration step becomes more and more complex and the convergence behaviour of (22) is extremely hard to determine. In principle, one could look for a termination condition. However, this is superfluous, since (22) becoming a finite series would necessarily diverge for $r \to \infty$, if one were to be found. In order for a finite series not to diverge, we need to modify the ansatz (22) by a factor compensating the r^n-terms for each n as $r \to \infty$. One possible solution is an exponential function e^{-dr} with d a positive constant. The modified ansatz now reads

$$\phi_{lc} = e^{-dr} r^{2l} \sum_{i=0}^{\infty} a_i (-1)^i r^i. \tag{24}$$

We again evaluate the derivatives of eq.(21):

$$\frac{3}{r}\partial_r \phi_{lc} = 3e^{-dr} r^{2l} \sum_{i=0}^{\infty} a_i (-1)^i \left(r^{i-2}(2l+i) - r^{i-1} d \right)$$

$$\partial_r^2 \phi_{lc} = e^{-dr} r^{2l} \sum_{i=0}^{\infty} a_i (-1)^i \left[r^{i-2}(2l+i)(2l+i-1) - 2r^{i-1}(2l+i)d + r^i d^2 \right].$$

Eq.(21) now reads

$$\sum_{i=0}^{\infty} a_i (-1)^i r^{2l} \left[r^{i-2} i(4l+i+2) - r^{i-1}(4l+2i+3)d \right.$$
$$\left. + r^i \left(\omega^2 + d^2 + \frac{c}{2} \right) + \frac{c}{2} \sum_{n=0}^{\infty} (-1)^n r^{n+i+1} \frac{n}{(n+1)!} \right] = 0.$$

In this equation the coefficients of $e^{-dr} r^m$ have to vanish for every m. The coefficient of $e^{-dr} r^{2l+i-2}$ is

$$a_i(-1)^i i(4l+i+2) + a_{i-1}(-1)^i (4l+2i+1)d + a_{i-2}(-1)^i \left(\omega^2 + d^2 + \frac{c}{2} \right)$$
$$+ \frac{c}{2} \sum_{j+n=i-3} a_j (-1)^{j+n} \frac{n}{(n+1)!} = 0$$

and thus

$$a_i = \frac{-1}{i(4l+i+2)} \left[a_{i-1}(4l+2i+1)d + a_{i-2} \left(\omega^2 + d^2 + \frac{c}{2} \right) \right.$$
$$\left. - \frac{c}{2} \sum_{n+j=i-3} a_j \frac{n}{(n+1)!} \right].$$

Just as above, the sum of previous coefficients makes it difficult to predict the convergence behaviour of the power series (24). Even though a truncation of this infinite series to a finite one by finding a termination condition would not lead to a divergence for large r in this case, there is no simple way of actually finding any such condition.

It is apparent that an exponential term in $u(r)$ is extremely hard to deal with when it comes to investigating the convergence behaviour of the resulting power series. Hence, it seems necessary to restrain from exponential terms (or any terms that must first be expanded into an infinite power series themselves before the evaluation) in the following.

4.2 A more simple u(r)

We again use the ansatz (24) for ϕ_{lc}, but for $u(r)$ we now choose

$$u(r) = \alpha + \beta r^{-1} + \gamma r^{-2} \tag{25}$$

with constant $\alpha \neq 0$, $\beta \neq 0$ and $\gamma \neq 0$. This modification leads to a field configuration that is neither in Landau gauge nor has a finite action and serves the sole purpose of giving some insight as to how solutions of eq.(12) with ϕ_{lc} of the form (24) might look like. Eq.(12) takes the form

$$\sum_{i=0}^{\infty} a_i(-1)^i e^{-dr} r^{2l} \Bigg[r^{i-2} \left(i(4l+1+2) + \frac{c\gamma}{2} \right) \\ -r^{i-1} \left((4l+2i+3)d - \frac{c\beta}{2} \right) + r^i \left(\omega^2 + d^2 + \frac{c\alpha}{2} \right) \Bigg] = 0.$$

Again, the coefficients of every power of r are required to equal zero. The coefficient of $e^{-dr} r^{2l+i-2}$ is

$$a_i(-1)^i \left(i(4l+i+2) + \frac{c\gamma}{2} \right) + a_{i-1}(-1)^i \left((4l+2i+1)d - \frac{c\beta}{2} \right) \\ + a_{i-2}(-1)^i \left(\omega^2 + d^2 + \frac{c\alpha}{2} \right) = 0.$$

If we require

$$d^2 = -\omega^2 - \frac{c\alpha}{2},$$

the term with a_{i-2} vanishes for every i, simplifying the recurrence formula, as now every coefficient a_i only depends on the previous one:

$$a_i = \frac{-1}{i(4l+i+2) + \frac{c\gamma}{2}} \left((4l+2i+1)d - \frac{c\beta}{2} \right) a_{i-1}.$$

We can now pick an arbitrary i at which the series (24) should get truncated if

$$\gamma = -\frac{2i}{c}(4l+i+2) = 0$$

$$\beta = \frac{2d}{c}(4l+2i+3) = 0.$$

If we truncate the series at $i > 0$, a_0 vanishes as $\gamma \neq 0$ and so does every subsequent coefficient except a_i, becoming a normalization constant $a_i = D$.

Therefore, only one single summand of (24) contributes to ϕ_{lc}. Truncating at $i = 0$ leads to $a_0 = 0$ as well, in which case (24) does not have any contributing summands at all.
As an example we choose $i = 2$ so that $a_2 = D$ and

$$u(r) = \alpha + \frac{2d}{c}(4l + 7)r^{-1} - \frac{16}{c}(l + 1)r^{-2} \tag{26}$$

$$\phi_{lc} = De^{-dr}r^{2(l+1)}$$
$$d^2 + \frac{c\alpha}{2} = -\omega^2.$$

The solutions ϕ_{lc} are normalizable if $d > 0$, setting a restriction on α as $\omega^2 < 0$. It is important to note that $u(r)$ must not depend on l. Thus, the modification (26) only works for one certain l that needs to be chosen beforehand. The exact same holds for c. Again, this ansatz was only intended to give some insight into the possible solutions of eq.(12).

4.3 Half-integer exponents

We analyze a modification with half-integer exponents of r in it:

$$u(r) = \frac{\alpha + \beta r^{5/2}}{\gamma + \eta r^{7/2}} \tag{27}$$

Herein, α, β, γ and η are constants with the only restriction that γ and η must not vanish at the same time. This time, we use yet another modification of (22) for ϕ_{lc}:

$$\phi_{lc} = e^{-dr}r^{2l}\sum_{i=0}^{\infty} a_i(-1)^i r^{i+j}. \tag{28}$$

Eq.(12) now reads

$$\sum_{i=0}^{\infty} e^{-dr}a_i(-1)^i r^{2l+j}\Bigg[r^{i-2}(i+j)(4l+i+j+2)\gamma - r^{i-1}(4l+2i+2j+3)d\gamma + r^i\left((\omega^2+d^2)\gamma + \frac{c\alpha}{2}\right) + r^{i+\frac{3}{2}}(i+j)(4l+j+i+2)\eta - r^{i+\frac{5}{2}}\left((4l+2i+2j+3)d\eta - \frac{c\beta}{2}\right) + r^{i+\frac{7}{2}}(\omega^2+d^2)\eta\Bigg] = 0.$$

Because of the half-integer exponents, there are now two conditions that have to hold, as both the coefficients of $e^{-dr}r^{2l+i+j-2}$ and $e^{-dr}r^{2l+i+j+\frac{3}{2}}$ have to vanish simultaneously for every i and thus

$$a_i(i+j)(4l+i+j+2)\gamma + a_{i-1}(4l+2i+2j+1)d\gamma + a_{i-2}\left((\omega^2+d^2)\gamma + \frac{c\alpha}{2}\right) = 0$$

$$a_i(i+j)(4l+i+j+2)\eta + a_{i-1}\left((4l+2i+2j+1)d\eta - \frac{c\beta}{2}\right) + a_{i-2}(\omega^2+d^2)\eta = 0.$$

The first step of the iteration is

$$a_0 j(4l + j + 2)\gamma = 0$$

$$a_0 j(4l + j + 2)\eta = 0.$$

As it is not allowed for γ and η to vanish simultaneously, we conclude that $a_0 = 0$. The following iteration steps are quite similar to the first one, as every subsequent coefficient $a_k = 0$ for $k < i$ up to the point i where

$$(i + j)(4l + i + j + 2) = 0$$

and thus $j = -i$ or $j = -4l - i - 2$. Consequently, we can make the corresponding coefficient a constant $a_i = D$. We again look for termination conditions, the most simple of which is to set $a_k = 0$ for $k > i$ as well[5]. To achieve this, we take a look at the next iteration step, already with $a_{i+1} = 0$:

$$D(4l + 2i + 2j + 3)d\gamma = 0$$

$$D\left((4l + 2i + 2j + 3)d\eta - \frac{c\beta}{2}\right) = 0.$$

We are left with no choice other than setting

$$\gamma = 0$$

and

$$\beta = \frac{2d\eta}{c}(4l + 2i + 2j + 3).$$

The next iteration step then reads

$$D\frac{\alpha c}{2} = 0$$

$$D(\omega^2 + d^2)\eta = 0.$$

The only way to make this work is

$$\alpha = 0$$

$$\omega^2 + d^2 = 0$$

as γ already vanishes and thus $\eta \neq 0$. This also guarantees that $d > 0$ because $\omega^2 < 0$. What remains of (27) is

$$u(r) = \frac{2d}{rc}(4l + 2i + 2j + 3)$$

[5] This whole procedure also works for $i = 0$, leading to $a_0 = D$ and $a_k = 0$ for $k > 0$.

and with the two possibilities of j:

$$u(r) = \frac{2d}{rc}(4l+3)$$

$$u(r) = -\frac{2d}{rc}(4l+1).$$

The corresponding eigenstates read

$$\phi_{lc} = De^{-dr}r^{2l}$$

$$\phi_{lc} = De^{-dr}r^{-2(l+1)}$$

$$d^2 = -\omega^2.$$

Unfortunately, the modifications (27) lead to field configurations that neither obey the Landau gauge condition (3) nor have a finite action. Additionally, they also contain l and c, just like the one treated in the preceding section and thus only work for one particular l and c as well. As for the solutions ϕ_{lc}, the first one is normalizable for every allowed l as $d > 0$, while the second one is normalizable only for $l < 3$.

4.4 Second-order polynomial as denominator of u(r)

The next modification to be investigated is

$$u(r) = \frac{\alpha}{\beta + \gamma r + \eta r^2} \tag{29}$$

with constant α, β, γ and η and the restriction that β, γ and η must not vanish at the same time. Again using the ansatz (28), eq.(12) now takes the form

$$\begin{aligned}\sum_{i=0}^{\infty} a_i(-1)^i e^{-dr} r^{2l+j} \Bigg[& r^{i-2}(i+j)(4l+i+j+2)\beta \\ & +r^{i-1}((i+j)(4l+i+j+2)\gamma - (4l+2i+2j+3))d\beta \\ & +r^i \left((i+j)(4l+i+j+2)\eta - (4l+2i+2j+3)d\gamma + (\omega^2+d^2)\beta + \frac{c\alpha}{2}\right) \\ & +r^{i+1}(-(4l+2i+2j+3)d\eta + (\omega^2+d^2)\gamma) + r^{i+2}(\omega^2+d^2)\eta\Bigg] = 0.\end{aligned}$$

The coefficient of $e^{-dr}r^{2l+i+j-2}$ having to vanish for every i leads to the equation

$$\begin{aligned} & a_i(i+j)(4l+i+j+2)\beta \\ & -a_{i-1}((i+j-1)(4l+i+j+1)\gamma - (4l+2i+2j+1)d\beta) \\ & +a_{i-2}\left((i+j-2)(4l+i+j)\eta - (4l+2i+2j-1)d\gamma + (\omega^2+d^2)\beta + \frac{c\alpha}{2}\right) \\ & -a_{i-3}\left(-(4l+2i+2j-3)d\eta + (\omega^2+d^2)\gamma\right) + a_{i-4}(\omega^2+d^2)\eta = 0.\end{aligned}$$

For simplification, we again set

$$\omega^2 + d^2 = 0.$$

As in the scenarios treated above, we look for a termination condition. In order for the series (28) to terminate at a point i, at the same time making the coefficient a_i a normalization constant $a_i = D$, each of the following requirements has to hold:

$$(i+j)(4l+i+j+2)\beta = 0$$

$$\gamma(i+j)(4l+i+j+2) - (4l+2i+2j+3)d\beta = 0$$

$$(i+j)(4l+i+j+2)\eta - (4l+2i+2j+3)d\gamma + \frac{c\alpha}{2} = 0$$

$$(4l+2i+2j+3)d\eta = 0.$$

One way to achieve this is to set $j = -i$ or $j = -4l - i - 2$, while $\beta = 0$, $\eta = 0$ and

$$\frac{\alpha}{\gamma} = \frac{2d}{c}(4l+2i+2j+3),$$

which leads to the exact same solutions as the ones treated in section 4.3. Another possibility is $\beta = 0$, $\gamma = 0$ and $j = -2l - i - \frac{3}{2}$. In this case,

$$\frac{\alpha}{\eta} = \frac{2}{c}\left(2l + \frac{3}{2}\right)\left(2l + \frac{1}{2}\right).$$

Now, the modification $u(r)$ and the solution ϕ_{lc} read

$$u(r) = \frac{(4l+3)(4l+1)}{2cr^2}$$

$$\phi_{lc} = De^{-dr}r^{-\frac{3}{2}}$$

$$d^2 = -\omega^2.$$

Again, $u(r)$ leads to a field configuration that neither is in Landau gauge nor has a finite action. Additionally, it once again depends on l and c and the solution ϕ_{lc} is non-normalizable.

4.5 A general approach

For the sake of completeness, we analyze a general modification $u(r)$ that can be expanded into a power series, despite knowing that its convergence behaviour is likely difficult to determine:

$$u(r) = \sum_{k=0}^{\infty} b_k r^k. \tag{30}$$

We now go back to using (24) for ϕ_{lc} and eq.(12) takes the form

$$\sum_{i=0}^{\infty} a_i(-1)^i e^{-dr} r^{2l} \Bigg[r^{i-2}(i(4l+i+2) - r^{i-1}(4l+2i+3)d$$

$$+ r^i(\omega^2 + d^2) + \frac{c}{2}\sum_{k=0}^{\infty} b_k r^{k+i} \Bigg] = 0.$$

As in the previous sections, the coefficient of r^m has to vanish for every m in order for this equation to hold. The coefficient of $e^{-dr}r^{2l+i-2}$ reads

$$a_i(-1)^i i(4l+i+2)+a_{i-1}(-1)^i(4l+2i+1)d+a_{i-2}(-1)^i(\omega^2+d^2) \\ +\frac{c}{2}\sum_{k+j=i-2} a_j b_k (-1)^j = 0.$$

As there is no simple way to find out whether or not this leads to a convergent series or to find any termination condition, the only thing left to do is to check under which conditions the Landau gauge-condition (3) holds for this modification:

$$\frac{\partial u(r)}{r\partial_r} = \sum_{k=0}^{\infty} b_k k r^{k-2} = \sum_{k=1}^{\infty} b_k k r^{k-2}$$

The limit $r \to 0$ looks like

$$\lim_{r\to 0}\sum_{k=1}^{\infty} b_k k r^{k-2} = \lim_{r\to 0} b_1 r^{-1} + 2b_2.$$

If we require $\frac{\partial u(r)}{r\partial_r}$ to be finite at every point, then

$$b_1 = 0.$$

4.6 A general rational approach

This is another modification we discuss just for the sake of completeness:

$$u(r) = \frac{\sum_{j=0}^{J} g_j r^j}{\sum_{k=0}^{K} h_k r^k}. \tag{31}$$

with $K > J$ to facilitate convergence as $r \to \infty$. Once again using the ansatz (24), eq.(12) reads[6]

$$\sum_{i=0}^{\infty} a_i(-1)^i e^{-dr} r^{2l}\Bigg[\sum_{k=0}^{K} h_k r^{i+k-2} i(4l+i+2) - \sum_{k=0}^{k} h_k r^{i+k-1}(4l+2i+3)d \\ + \sum_{k=0}^{K} h_k r^{i+k}(\omega^2+d^2) + \frac{c}{2}\sum_{j=0}^{J} g_j r^{j+i}\Bigg] = 0.$$

The coefficient of $e^{-dr}r^{2l+i-2}$ now takes the form

$$\sum_{m+k=i} a_m h_k (-1)^m r^{m+k-2} m(4l+m+2) \\ - \sum_{m+k=i-1} a_m h_k (-1)^m r^{m+k-1}(4l+2m+1)d \\ + \sum_{m+k=i-2} a_m h_k (-1)^m r^{m+k}(\omega^2+d^2) \\ +\frac{c}{2}\sum_{j+m=i-2} a_m g_j (-1)^m r^{j+m} = 0.$$

[6] Note that in the summations $k \leq K$ and $j \leq J$.

Again, we check whether or not this modification leads to a field configuration that is in Landau gauge:

$$\frac{\partial u(r)}{r\partial_r} = \frac{1}{r\left(\sum_{k=0}^{K} h_k r^k\right)^2}\left[\sum_{j=0}^{J} g_j j r^{j-1}\sum_{k=0}^{K} h_k r^k - \sum_{j=0}^{J} g_j r^j \sum_{k=0}^{K} h_k k r^{k-1}\right]$$

and in the limit $r \to 0$

$$\lim_{r\to 0}\frac{\partial u(r)}{r\partial_r} = \frac{1}{h_0^2}(g_1 h_0 - g_0 h_1) r^{-1}.$$

If $\frac{\partial u(r)}{r\partial_r}$ is required to be finite for every r, then

$$g_1 h_0 - g_0 h_1 = 0.$$

5 Summary and outlook

The instanton field configuration (6) has been modified to look like (7). Various different $u(r)$ have been investigated in order to find normalizable eigenstates of the Faddeev-Popov operator (4) that host negative eigenvalues. For a constant $u(r)$, the calculation has been done algebraically, while for the remaining modifications power series expansions have been used.

While some solutions have been found, none of them fulfill all the requirements imposed on them: The constant $u(r)$ leads to a field configuration that is in Landau gauge, but does not have a finite action, while the eigenstates are non-normalizable. As for the power series expansions, there are in fact some normalizable eigenstates, the corresponding field configurations of which are not in Landau-gauge, though, and neither have a finite action. Some general considerations have been made, which could hypothetically lead to the desired results but are too complex to investigate without any further restrictions.

A lot of the work comes down to trial and error and since we only treated a few selected $u(r)$ in this thesis, we have left an abundance of possibilities unconsidered. However, this also opens up this topic for further research in the future.

References

[1] M. Bohm, A. Denner, and H. Joos. *Gauge theories of the strong and electroweak interaction*. Teubner, Stuttgart, 2001.

[2] A. Maas. Gauge bosons at zero and finite temperature. *Phys. Rept.*, **524**:203, 2013.

[3] V. N. Gribov. Quantization of Nonabelian Gauge Theories. *Nucl. Phys. B*, **139**:1, 1978.

[4] I. M. Singer. Some Remarks on the Gribov Ambiguity. *Commun. Math. Phys.*, **60**:7, 1978.

[5] D. Zwanziger. Nonperturbative Landau gauge and infrared critical exponents in QCD. *Phys. Rev. D*, **65**:094039, 2002.

[6] A. Maas. On the spectrum of the Faddeev-Popov operator in topological background fields. *Eur. Phys. J. C*, **48**:179, 2006.

[7] M. Abramowitz and I. A. Stegun. *Handbook of Mathematical Functions - With Formulas, Graphs, and Mathematical Tables*. Courier Corporation, 1964.