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Bachelor thesis (BSc)

Search for Gribov copies outside the first Gribov region in $SU(2)$

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Abstract

In non-Abelian gauge theories gauge fixing cannot be done by introducing only local conditions. A common way to impose a further global restriction is to restrict the residual gauge orbit to the first Gribov region. Another way to deal with the problem of gauge fixing in principle is to take all Gribov copies of each Gribov region and take their average. In this thesis Gribov copies are being searched outside the first Gribov region for the vacuum in $SU(2)$ Yang-Mills theory by looking for negative eigenvalues of the Fadeev-Popov operator.

By constructing gauge transformations of the vacuum a set of transformations has been found that remains in Landau gauge. The case of gauge transformations of vacuum with only a radial component which only depends on the radius in hyper-spherical coordinates is further examined. Using separation of variables the eigenproblem of the Fadeev-Popov operator was reduced to a coupled system of ordinary linear differential equations for this case. A power series expansion was used and the coefficients of this formal power series determined. Although convergence for these power series was not shown a combination of these coefficients has been shown to converge absolutely. An idea how this can be used to show convergence for the power series is given in addition.

Contents

1. Introduction

Gauge theories are one of the most important parts of modern physics. One essential characteristic is the existence of transformations which change the involved potentials and wavefunctions without affecting the observables. These transformation are called gauge transformations. All observables are gauge independent.
Not every quantity that is useful in performing calculations is however gauge independent and it is therefore necessary to fix a gauge. In Abelian gauge theories it is possible to fix a gauge by introducing only local constraints. In non-Abelian gauge theories however this is no longer possible. Local gauge conditions cannot fix a gauge completely due to the structure of non-Abelian gauge theories and the remaining solutions are called Gribov copies [1].
One way to introduce a further global restriction is by demanding that the Fadeev-Popov operator has a positive semi-definite spectrum, i.e. all its eigenvalues are either positive or zero. The region in which this operator is positive semi-definite is called the first Gribov region [1].
If all Gribov copies were found this could be used to average over all Gribov copies for performing calculations. The aim of this thesis is to find Gribov copies outside the first Gribov region, i.e. gauge transfomations where the Fadeev-Popov operator has negative eigenvalues starting with the simplest case: vacuum in the simplest non-Abelian gauge theory $SU(2)$.

2. Gauge theories

This section follows closely [2].

2.1. Gauge invariance in electromagnetism

The theory in which gauge invariance was first observed was the theory of electromagnetism. The concept has been extended to include more general theories. [3] It seems reasonable to introduce the concept of gauge invariance and gauge theories in the context of electromagnetism first and then move on to quantum mechanics. The laws describing electromagnetism are known as Maxwell's equations: [1].

$$\begin{aligned} \boldsymbol{\nabla} \boldsymbol{E} &= \rho_{em} & \boldsymbol{\nabla} \times \boldsymbol{E} &= -\partial_t \boldsymbol{B} \\ \boldsymbol{\nabla} \boldsymbol{B} &= 0 & \boldsymbol{\nabla} \times \boldsymbol{B} &= \boldsymbol{j_{em}} + \partial_t \boldsymbol{E} \end{aligned} \tag{1}$$

Here ρ_{em} denotes the electromagnetic charge density and j_{em} denotes the electromagnetic current density. The fields $\boldsymbol{B}$ and $\boldsymbol{E}$ can be expressed in terms of a vector potential $\boldsymbol{A}$ and a scalar potential V:

$$\boldsymbol{B} = \boldsymbol{\nabla} \times \boldsymbol{A} \qquad \boldsymbol{E} = -\boldsymbol{\nabla} V - \partial_t \boldsymbol{A} \tag{2}$$

[1] In this thesis natural units $c = \hbar = 1$ are used.

These potentials are however not unique because a transformation of the potentials of the form $\boldsymbol{A} \to \boldsymbol{A'} = \boldsymbol{A} + \boldsymbol{\nabla}\chi(\boldsymbol{x},t)$, $V \to V' = V - \partial_t\chi(\boldsymbol{x},t)$ does not change the observable fields $\boldsymbol{B}$ and $\boldsymbol{E}$ and therefore the transformed potentials represent the same physical reality.
Here $\chi(\boldsymbol{x},t)$ is an arbitrary differentiable function. A transformation of this kind is called a gauge transformation and the theory is said to exhibit a gauge freedom. The theory of electromagnetism is relativistic and can therefore be written using relativistic notation. Using the metric $(+,-,-,-)$ both potentials can be combined to form the four potential A^μ. Similarly the densities and the derivatives can be combined.

$$A^\mu = (V, \boldsymbol{A}) \qquad \partial^\mu = (\partial_t, -\boldsymbol{\nabla}) \qquad j^\mu_{em} = (\rho_{em}, j_{em}) \tag{3}$$

Maxwell's equations can now be written as:

$$\partial_\mu F^{\mu\nu} = j^\nu_{em} \tag{4}$$
$$F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu \tag{5}$$

A gauge transformation is then given by:

$$A^\mu \to A'^\mu = A^\mu - \partial^\mu\chi \tag{6}$$

2.2. Gauge invariance in quantum mechanics

The classical Hamiltonian for a charged particle of charge q and mass m in an electromagnetic field is:

$$\boldsymbol{H} = \frac{1}{2m}\left(\boldsymbol{p} - q\boldsymbol{A}\right)^2 + qV \tag{7}$$

The Schrödinger equation for this particle is

$$\left(\frac{1}{2m}\left(-i\boldsymbol{\nabla} - q\boldsymbol{A}\right)^2 + qV\right)\psi(\boldsymbol{x},t) = i\partial_t\psi(\boldsymbol{x},t) \tag{8}$$

It has been previously shown that the electromagnetic fields do not change upon a gauge transformation. An interesting question is how the wavefunction should transform if the Schrödinger equation is required not to change upon a simultaneous transformation of the potentials and the wavefunction.

The result will not be derived but only stated (9). It is straightforward to check by transforming all affected quantities in (8). Maxwell's equations remain gauge invariant in quantum mechanics under the following simultaneous transformations:

$$\psi(\boldsymbol{x},t) \to \psi'(\boldsymbol{x},t) = e^{iq\chi(\boldsymbol{x},t)}\psi(\boldsymbol{x},t) \tag{9}$$
$$\boldsymbol{A} \to \boldsymbol{A'} = \boldsymbol{A} + \boldsymbol{\nabla}\chi(\boldsymbol{x},t) \tag{10}$$
$$V \to V' = V - \partial_t\chi(\boldsymbol{x},t) \tag{11}$$

The transformed wavefunction gains a phase factor which is dependent on space and time. The wavefunction is said to exhibit local phase invariance.

2.3. Gauge principle

In the previous section it was demanded that the wavefunction should not change upon a gauge transformation of the potentials. This argument can be reversed by demanding that the wavefunction is invariant under a local phase transformation. It can be shown that this is not possible for a free theory. It is required to introduce a field whose potentials transform exactly like (10) and (11). This is the basis of the gauge principle: Demanding invariance under a local transformation for the wavefunction leads to the appearence of compensating gauge fields.
In the case of the demanded local transformation being a local phase transformation of the form (9) the compensating gauge fields are the electric and the magnetic field. Note that local phase transformations commute because the original wavefunction is multiplied by a complex number. It is therefore called an commutative or Abelian gauge transformation. All possible transformations form the group of one-dimensional unitary matrices $U(1)$.

2.4. Non-Abelian gauge theories

The idea of demanding invariance of a theory under a local transformation of the wavefunction can be further generalized by considering more than one wavefunction. This is motivated by the fact that in the case of a set of degenerate states a linear combination will be as well an admissible solution. The simplest case is using a linear combination of two wavefunctions with the complex coefficients α,β,γ and δ of the original wavefunction.

$$\psi_1 \rightarrow \psi_1' = \alpha\psi_1 + \beta\psi_2 \tag{12}$$

$$\psi_2 \rightarrow \psi_2' = \gamma\psi_1 + \delta\psi_2 \tag{13}$$

The two wavefunctions can then be understood as the components of one object $\psi^{(1/2)}$:

$$\psi^{(1/2)} \equiv \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} \tag{14}$$

The equations (12) and (13) can now be written as

$$\begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} \rightarrow \begin{pmatrix} \psi_1' \\ \psi_2' \end{pmatrix} = h \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} \tag{15}$$

where h is a complex 2×2 matrix. If the physics should not change upon a transformation of this kind the probabilities have to be preserved. Therefore the matrix h is required to be unitary. From the properties of the determinant it follows that $|\det(h)| = 1$ [2]. These transformations form the group of 2×2 unitary matrices $U(2)$ and can therefore be regarded as the two-dimensional generalization of the electromagnetic case. The cases of $U(n)$ can be constructed similarly by including more wavefunctions in the linear combination. If the determinant of h is equal to one the matrix is said to be special

[2] $|\det U|^2 = \det U \overline{\det U} = \det U \det U^\dagger = \det\left(UU^\dagger\right) = det\boldsymbol{I} = 1$

orthogonal and all matrices fullfilling this requirement form the special orthogonal group $SU(2)$. A unitary matrix has in general 4 real valued parameters, a special unitary matrix has 3 because of the extra condition for the determinant to be equal to one.
Note that these matrices in general do not commute. Any $SU(2)$ matrix of this form can be written as $h = \exp\left(\frac{i\sigma^a\tilde{\phi}^a}{2}\right) = \exp\left(i\tau^a\tilde{\phi}^a\right)$, where $\tau^a = \frac{1}{2}\sigma^a$ and σ^a are the Pauli matrices

$$\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma^2 = \begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix}, \quad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \tag{16}$$

i is the imaginary unit, $a = 1, 2, 3$ and $\tilde{\phi}^a$ are real valued. The exponential function is defined as a power series of the matrix $i\tau^a\tilde{\phi}^a$. [3] Up to now the transformations given by (15) are global transformations. In order to obtain a gauge field it is required that this phase transformation invariance is a local transformation with respect to spacetime, $\tilde{\phi}^a$ are therefore not constants but functions of spacetime.

With $\boldsymbol{I}$ denoting the 2×2 identity matrix the product of any two Pauli matrices is $\sigma^a\sigma^b = \boldsymbol{I}\delta^{ab} + i\epsilon^{abc}\sigma^c$. In $SU(2)$ gauge theory a gauge field A_μ appears that transforms according to (17) to guarantee local phase invariance [4]:

$$A_\mu \to A'_\mu = -ih^{-1}\left(iA_\mu - \partial_\mu\right)h \tag{17}$$

2.5. Gauge fixing and Gribov copies

Not every useful quantity is however gauge independent and it is therefore necessary to fix a gauge for many calculations. In Abelian gauge theories it is possible to fix a gauge by introducing local constraints for example the Lorenz or Landau gauge $\partial_\mu A_\mu = 0$. [4] Other gauge conditions exist but not all of them fix the gauge completely [3]. In non-Abelian gauge theories this is however no longer possible. Local gauge conditions cannot fix a gauge completely due to the structure of non-Abelian gauge theories and the remaining solutions to the Landau gauge condition are called Gribov copies. [1] One way to introduce a further restriction is by demanding that the Fadeev-Popov operator has a positive semi-definite spectrum, i.e. all its eigenvalues are either positive or zero. The region in which this operator is positive semi-definite is called the first Gribov region. It has been proven that there is always at least one Gribov copy in the first Gribov region [1]. The Fadeev-Popov operator in $SU(2)$ takes the form [5]:

$$M^{ab} = -\partial_\mu\left(\partial_\mu\delta^{ab} + g\epsilon^{abc}\partial_\mu A^c_\mu\right) \tag{18}$$

Here ϵ^{abc} is the Levi-Civita sybmol with $\epsilon^{123} = 1$ and $A_\mu = \tau^c A^c_\mu$. On the boundary of the first Gribov region the Fadeev-Popov operator has zero eigenvalues, this boundary is called the Gribov horizon. Further Gribov regions are always separated by a Gribov

[3] Note that in this thesis the Einstein summation convention is used.
[4] All calculations in this thesis are done in 4-dimensional Euclidean space.

horizon, each of them has one more negative eigenvalue, i.e. the second Gribov region has one negative eigenvalue, the third Gribov region has two negative eigevalues and so on. It has not yet been proven that there always exists a Gribov copy for every region, but this is expected to be true [1].

In this thesis Gribov copies are being searched outside the first Gribov region, i.e. gauge transfomations of the vacuum where the Fadeev-Popov operator has negative eigenvalues. If all Gribov copies for every Gribov region were found this could be used to average over all Gribov copies for performing calculations. Since not much is known explicitly about Gribov copies outside the first Gribov region it is reasonable to start with the simplest case: vacuum in the simplest non-Abelian gauge theory $SU(2)$.

3. Gradient, divergence and Laplacian in hyperspherical coordinates

Later on the discussion of potentials will be restricted to radially symmetric problems in 4-dimensional Euclidean space. Therefore the hyperspherical coordinate system is a convenient choice.

$$\begin{aligned} r_\mu &= (x, y, z, t)^T \\ &= (r\cos\eta, r\cos\theta\sin\eta, r\cos\phi\sin\theta\sin\eta, r\sin\phi\sin\theta\sin\eta)^T \end{aligned} \tag{19}$$

Here ϕ ranges in $[0, 2\pi)$ and θ and η range in $[0, \pi)$. The action of the differential operators in hyperspherical coordinates can then be obtained [6]. The unit vectors can be written as:

$$\boldsymbol{e_{x_i}} = \frac{\partial r_\mu}{\partial x_i}\left|\frac{\partial r_\mu}{\partial x_i}\right|^{-1} \tag{20}$$

Note that in general unit vectors of a given coordinate system are not constant. The gradient of a scalar function in orthogonal coordinates is:

$$\begin{aligned} \boldsymbol{\nabla} f &= (\boldsymbol{e_x}\partial_x + \boldsymbol{e_y}\partial_y + \boldsymbol{e_z}\partial_z + \boldsymbol{e_t}\partial_t)\, f && (21) \\ &= \sum_{i=1}^{4} \boldsymbol{e_{x_i}}\left|\frac{\partial r_\mu}{\partial x_i}\right|^{-1}\partial_{x_i} f && (22) \end{aligned}$$

In the case of 4-dimensional spherical coordinates an explicit expression for the gradient can now be obtained by substituting (19) into (22).

$$\boldsymbol{\nabla} f = \left(\boldsymbol{e_r}\partial_r + \boldsymbol{e_\eta}\frac{1}{r}\partial_\eta + \boldsymbol{e_\theta}\frac{1}{r\sin\eta}\partial_\theta + \boldsymbol{e_\phi}\frac{1}{r\sin\theta\sin\eta}\partial_\phi\right) f \tag{23}$$

A general vector field in hyperspherical coordinates has the form:

$$\boldsymbol{A} = \boldsymbol{e_r}A_r + \boldsymbol{e_\eta}A_\eta + \boldsymbol{e_\theta}A_\theta + \boldsymbol{e_\phi}A_\phi \tag{24}$$

The formal scalar product of the del-operator ∇ in hyperspherical coordinates and the vector field $\boldsymbol{A}$ can be interpreted as the divergence of the vector field.

$$\boldsymbol{\nabla}\cdot\boldsymbol{A} = \frac{1}{r^3}\partial_r(r^3A_r) + \frac{1}{r\sin^2\eta}\partial_\eta(\sin^2\eta A_\eta) + \frac{1}{r\sin\theta\sin\eta}\left(\partial_\theta(\sin\theta A_\theta) + \partial_\phi A_\phi\right) \tag{25}$$

The Laplacian acting on a scalar function is equal to the divergence of the gradient of that function (23).

$$\Delta f = \frac{1}{r^3}\partial_r(r^3\partial_r)f + \frac{1}{r^2\sin^2\eta}\left(\partial_\eta(sin^2\eta\partial_\eta) + \frac{1}{\sin\theta}\partial_\theta(\sin\theta\partial_\theta) + \frac{1}{\sin^2\theta}\partial_{\phi\phi}\right)f \tag{26}$$

4. The Fadeev-Popov operator in vacuum

4.1. Gauge transformations of vacuum

In vacuum the potential A_μ in the untransformed case is equal to zero. The Fadeev-Popov operator (18) is then only the negative Laplacian times a unit matrix in color space. In Cartesian coordinates the eigenfunctions of the Laplacian are plane waves $C^a\exp(ik_\mu r_\mu)$. The eigenvalues of the Fadeev-Popov operator in vacuum are then $k_\mu k_\mu > 0$. The spectrum is therefore postive. According to equation (17) the potentials described in the following way are gauge transforms of the vacuum potential:

$$A_\mu = i(\partial_\mu h)h^{-1} \tag{27}$$

Differentiating h and multiplying it by its inverse in the form of a matrix exponential proves to be rather difficult since the matrices involved do not commute. It is therefore reasonable to express h in a way that is easier to differentiate with respect to spacetime. First note that

$$h = \exp\left(i\tau^a\tilde{\phi}^a\right) = \exp\left(i\tau^a \underbrace{\frac{\tilde{\phi}^a}{\sqrt{\tilde{\phi}^b\tilde{\phi}^b}}}_{=\phi^a} \underbrace{\sqrt{\tilde{\phi}^b\tilde{\phi}^b}}_{=\varphi}\right) = \exp(i\tau^a\phi^a\varphi) \tag{28}$$

where the sum over all squares of $\phi^a\phi^a$ is now equal to one. Differentiating this expression gives $\phi^a\partial_\mu\phi^a = 0$. The form of h as a complex exponential of a matrix can now be decomposed into a sine and a cosine [7].

$$h = \exp(i\tau^a\phi^a\varphi) = \boldsymbol{I}\cos\left(\frac{\varphi}{2}\right) + i\sigma^a\phi^a\sin\left(\frac{\varphi}{2}\right) \tag{29}$$

$$h^{-1} = \exp(-i\tau^a\phi^a\varphi) = \boldsymbol{I}\cos\left(\frac{\varphi}{2}\right) - i\sigma^a\phi^a\sin\left(\frac{\varphi}{2}\right) \tag{30}$$

$$\partial_\mu h = \boldsymbol{I}\sin\left(\frac{\varphi}{2}\right)\left(\frac{-1}{2}\right)\partial_\mu\varphi + i\sigma^a\left(\frac{1}{2}\phi^a\cos\left(\frac{\varphi}{2}\right)\partial_\mu\varphi + \sin\left(\frac{\varphi}{2}\right)\partial_\mu\phi^a\right) \tag{31}$$

The gauge transformation of the vacuum multiplied by i can now be written as:

$$(\partial_\mu h)h^{-1} = \left(\boldsymbol{I}\sin\left(\frac{\varphi}{2}\right)\left(\frac{-1}{2}\right)\partial_\mu\varphi + i\sigma^a\left(\frac{1}{2}\phi^a\cos\left(\frac{\varphi}{2}\right)\partial_\mu\varphi + \sin\left(\frac{\varphi}{2}\right)\partial_\mu\phi^a\right)\right) \cdot \left(\boldsymbol{I}\cos\left(\frac{\varphi}{2}\right) - i\sigma^b\phi^b\sin\left(\frac{\varphi}{2}\right)\right) \tag{32}$$

Expanding the first pair of brackets and using the trigonometric double-angle identity $\cos(x)\sin(x) = \frac{1}{2}\sin(2x)$ gives:

$$(\partial_\mu h)h^{-1} = -\boldsymbol{I}\left(\frac{1}{4}\right)\sin(\varphi)\partial_\mu\varphi + \frac{i}{2}\sigma^b\phi^b\sin^2\left(\frac{\varphi}{2}\right)\partial_\mu\varphi + i\sigma^a\left(\frac{1}{2}\phi^a\cos^2\left(\frac{\varphi}{2}\right)\partial_\mu\varphi + \frac{1}{2}\sin(\varphi)\partial_\mu\phi^a\right) + \sigma^a\sigma^b\left(\frac{1}{2}\phi^a\cos\left(\frac{\varphi}{2}\right)\partial_\mu\varphi + \sin\left(\frac{\varphi}{2}\right)\partial_\mu\phi^a\right)\left(\phi^b\sin\left(\frac{\varphi}{2}\right)\right)$$

This result can be further simplified. The terms involving $\sin^2\left(\frac{\varphi}{2}\right)$ and $\cos^2\left(\frac{\varphi}{2}\right)$ combine into one term, since $\sin^2 x + \cos^2 x = 1$. The product of the Pauli matrices is known and the brackets in the last term can be expanded and the double-angle identity can be used again:

$$(\partial_\mu h)h^{-1} = -\boldsymbol{I}\left(\frac{1}{4}\right)\sin(\varphi)\partial_\mu\varphi + \frac{i}{2}\sigma^a\phi^a\partial_\mu\varphi + \frac{i}{2}\sin(\varphi)\sigma^a\partial_\mu\phi^a + \left(\boldsymbol{I}\delta^{ab} + i\epsilon^{abc}\sigma^c\right)\left(\frac{1}{4}\phi^a\phi^b\sin(\varphi)\partial_\mu\varphi + \sin^2\left(\frac{\varphi}{2}\right)(\partial_\mu\phi^a)\phi^b\right)$$

Expanding the last pair of brackets and using $\phi^a\phi^a = 1$ and $\phi^a\partial_\mu\phi^a = 0$ gives:

$$(\partial_\mu h)h^{-1} = -\boldsymbol{I}\left(\frac{1}{4}\right)\sin(\varphi)\partial_\mu\varphi + \frac{i}{2}\sigma^a\phi^a\partial_\mu\varphi + \frac{i}{2}\sin(\varphi)\sigma^a\partial_\mu\phi^a + \boldsymbol{I}\left(\frac{1}{4}\right)\sin(\varphi)\partial_\mu\varphi + i\epsilon^{abc}(\partial_\mu\phi^a)\phi^b\sigma^c\sin^2\left(\frac{\varphi}{2}\right) + \frac{1}{4}\sin(\varphi)\partial_\mu\varphi\epsilon^{abc}\phi^a\phi^b\sigma^c$$

The last term vanishes because $\epsilon^{abc}\phi^a\phi^b\sigma^c = -\epsilon^{bac}\phi^b\phi^a\sigma^c = -\epsilon^{abc}\phi^a\phi^b\sigma^c = 0$. Using the trigonometry identity $\sin^2(x) = \frac{1}{2}(1 - \cos(2x))$ the gauge transformed potential is:

$$A_\mu = i(\partial_\mu h)h^{-1} = -\tau^a\phi^a\partial_\mu\varphi - \tau^a(\partial_\mu\phi^a)\sin(\varphi) + (cos(\varphi) - 1)\,\epsilon^{abc}(\partial_\mu\phi^a)\phi^b\tau^c. \tag{33}$$

In order to obtain the Fadeev-Popov operator the projections of the potential onto the matrices τ^c is needed. These are easily obtained by factoring out τ^c. The index in the first two terms can be relabeled because it is only a summation index.

$$A_\mu = \tau^c A_\mu^c \tag{34}$$

$$\begin{aligned} &= i(\partial_\mu h)h^{-1} \\ &= -\tau^c\phi^c\partial_\mu\varphi - \tau^c(\partial_\mu\phi^c)\sin(\varphi) + (cos(\varphi) - 1)\,\epsilon^{abc}(\partial_\mu\phi^a)\phi^b\tau^c \\ &= \tau^c\left(-\phi^c\partial_\mu\varphi - (\partial_\mu\phi^c)\sin(\varphi) + (cos(\varphi) - 1)\,\epsilon^{abc}(\partial_\mu\phi^a)\phi^b\right) \end{aligned} \tag{35}$$

$$A^c_\mu = -\,\phi^c\partial_\mu\varphi - (\partial_\mu\phi^c)\sin(\varphi) + (cos(\varphi) - 1)\,\epsilon^{abc}(\partial_\mu\phi^a)\phi^b \tag{36}$$

From now on the discussion will be restricted to potentials that fulfill the Landau gauge condition (37). The components A^c_μ are real and therefore their derivative has to vanish for every single component in order to be in Landau gauge.

$$\partial_\mu A_\mu = 0 \tag{37}$$
$$\partial_\mu A^a_\mu = 0 \tag{38}$$

Consider now a gauge transformation specified by the three real valued functions $\tilde{\phi}^a$ which all differ only by a real factor u^a. This choice of functions simplifies the potential even more.

$$\tilde{\phi}^1 = -\,u^1\tilde{\phi}\ ,\ \tilde{\phi}^2 = -u^2\tilde{\phi}\ ,\ \tilde{\phi}^3 = -u^3\tilde{\phi} \tag{39}$$

The relations between $\tilde{\phi}$, ϕ and φ are given in equation (28).

$$\varphi = \sqrt{\tilde{\phi}^a\tilde{\phi}^a} = \sqrt{(u^1)^2 + (u^2)^2 + (u^3)^2}\tilde{\phi} \tag{40}$$
$$\partial_\mu\varphi = \sqrt{(u^1)^2 + (u^2)^2 + (u^3)^2}\partial_\mu\tilde{\phi} \tag{41}$$
$$\phi^a = \frac{-u^a}{\varphi} = \frac{-u^a}{\sqrt{(u^1)^2 + (u^2)^2 + (u^3)^2}} \tag{42}$$
$$\partial_\mu\phi^a = 0 \tag{43}$$
$$A^c_\mu = -\phi^c\partial_\mu\varphi = u^c\partial_\mu\tilde{\phi} \tag{44}$$

Now the Landau gauge condition is enforced which leads to the conclusion that the gauge transformation specified by multiples of $\tilde{\phi}$ is in Landau gauge if it is a solution to Laplace's equation in four dimensions.

$$\partial_\mu A^c_\mu = u^c\Delta\tilde{\phi} \overset{!}{=} 0 \rightarrow \Delta\tilde{\phi} = 0 \tag{45}$$

Now the function $\tilde{\phi(r)}$ is considered which only has radially dependence. The gauge transformation specified by this function is required to be in Landau gauge. Therefore it should be a solution to Laplace's equation. In 4-dimensional hyperspherical coordinates the Laplacian is given by equation (26).

$$\tilde{\phi}^c(r_\mu) = -\,u^c\tilde{\phi}(r) \tag{46}$$
$$\Delta\tilde{\phi}(r) = \frac{1}{r^3}\partial_r\left(r^3\tilde{\phi}(r)\right) = 0 \tag{47}$$

This is now an ordinary linear differential equation of second order with known solution from which the components of the potential can be obtained according to (44).

$$\tilde{\phi}^c = u^c\frac{1}{2r^2} + \text{const.} \tag{48}$$

$$A_\mu^c = -\, u^c \partial_\mu \tilde{\phi} = \frac{u^c}{r^3} \boldsymbol{e_r} \tag{49}$$

In Landau gauge the Fadeev-Popov (18) operator is:

$$M^{ab} = -\delta^{ab} \Delta - g \epsilon^{abc} A_\mu^c \partial_\mu \tag{50}$$

This is the only non constant potential of the form $\boldsymbol{e_r} A_r^c(r)$ in Landau gauge. This can be seen by solving $\partial_\mu \boldsymbol{e_r} A_r^c(r)$. The eigenequation for an eigenfunction ϕ^a to an eigenvalue λ in Landau gauge is:

$$M^{ab} \phi^b = \lambda \delta^{ab} \phi^b \tag{51}$$

$$(\Delta + \lambda)\, \phi^a + g \epsilon^{abc} A_\mu^c \partial_\mu \phi^b = 0 \tag{52}$$

Consider now the case where the potential has only a radial component which is furthermore only a function of the radius.

$$A_\mu^c = A_r^c \boldsymbol{e_r} \tag{53}$$

In the case of a potential which has only r-dependence it seems justifiable to make an ansatz of the following kind: Assume that the eigenfunctions can be written as a product of a radial function and a function which is dependent on the angular coordinates. Assume furthermore that the angular part of the three eigenfunctions is always the same.

$$\phi^a = R^a(r) Y(\eta, \theta, \varphi) = R^a Y \tag{54}$$

In appendix A it is shown that assuming all R^a to be equal is not a suitable choice for this problem and when looking for solutions to negative eigenvalues. Additionally an extra condition of the form $\epsilon^{abc} u^c \phi^b \neq 0$ for every a arises. If this expression was true then one of the differential equations would be equal to the untransformed case and would therefore lead to a positive spectrum. According to equation (26) the Laplacian can be decomposed in a term which only acts on the radial part of a function and a part which only acts on the angular part of that function times a factor $\frac{1}{r^2}$.

$$\Delta = \Delta_r + \frac{1}{r^2} \Delta_{\eta\theta\varphi} \tag{55}$$

Rewriting the eigenequation yields:

$$\left(\Delta_r + \frac{1}{r^2} \Delta_{\eta\theta\varphi} + \lambda \right) R^a Y + g \epsilon^{abc} A_\mu^c \partial_\mu \left(R^b Y \right) = 0 \tag{56}$$

Consider now the last term for the case of the potential only having a radial component with only r-dependence. The scalar product with the eigenfunctions vanishes for the angular part because the gradient of the angular component has no radial component.

$$A_\mu^c \partial_\mu \left(R^b Y \right) = A_r^c \boldsymbol{e_r} \left(Y \partial_\mu R^b + R^b \partial_\mu Y \right) = A_r^c \left(Y \underbrace{\boldsymbol{e_r} \partial_\mu R^b}_{= \partial_r R^b} + R^b \underbrace{\boldsymbol{e_r} \partial_\mu Y}_{=0} \right) = Y A_r^c \partial_r R^b \tag{57}$$

The eigenequation now reads:

$$\left(\Delta_r + \frac{1}{r^2}\Delta_{\eta\theta\varphi} + \lambda\right) R^a Y + g\epsilon^{abc} Y A_r^c \partial_r R^b = 0 \tag{58}$$

Dividing by $R^a Y$ and multiplying both sides by r^2 yields:

$$\frac{r^2 \Delta_r R^a}{R^a} + \lambda r^2 + \frac{r^2}{R^a} g\epsilon^{abc} A_r^c \partial_r R^b = -\frac{\Delta_{\eta\theta\varphi} Y}{Y} \tag{59}$$

The equation on the left-hand side depends only on the radius while the right-hand side depends on the angular coordinates. The equation has to hold true for every possible value of $(r, \eta, \theta, \varphi)$ and therefore both sides must be equal to a constant κ . The second order linear partial differential equation in 4 variables decouples in a second order linear ordinary differential equation and a second order linear partial differential equation in the three agular coordinates.

$$\frac{r^2 \Delta_r R^a}{R^a} + \lambda r^2 + \frac{r^2}{R^a} g\epsilon^{abc} A_r^c \partial_r R^b = -\frac{\Delta_{\eta\theta\varphi} Y}{Y} = \kappa \tag{60}$$

First consider the equation involving the angular part. By multiplying both sides by Y it is easy to see, that this is the eigenequation of the angular part of the Laplacian $\Delta_{\eta\theta\varphi}$ with the eigenvalues $-\kappa$.

$$-\frac{\Delta_{\eta\theta\varphi} Y}{Y} = \kappa \tag{61}$$

$$\Delta_{\eta\theta\varphi} Y = -\kappa Y \tag{62}$$

The eigenfunctions to this eigenequation are known as spherical harmonics. The eigenvalues to this problem are known [8]:

$$\kappa = l(l+2) \tag{63}$$

Here l is a positive integer or zero. The problem has therefore been reduced to a system of three coupled ordinary linear differential equations of second order since $\Delta_r = \frac{1}{r^3}\partial_r(r^3\partial_r)$.

$$\frac{1}{r}\partial_r(r^3\partial_r R^a) + (\lambda r^2 - \kappa)R^a + r^2 g\epsilon^{abc} A_r^c \partial_r R^b = 0 \tag{64}$$

Inserting the potential $A_r^c = u^c \frac{1}{r^3}$ and multiplying by r gives:

$$\left(r^3\lambda - r\kappa\right) R^a + \partial_r(r^3\partial_r R^a) + g\epsilon^{abc} u^c \partial_r R^b = 0 \tag{65}$$

4.2. Power series expansion

One way to solve such a system of differential equations (65) is by expanding the solution R^a as a formal power series. Substituting this assumed power series into the differential equation gives an infinite set of systems of linear equations which can be solved to yield the coefficient of the power series. In the last step one has to check, if the series converges. Otherwise the formal power series does not exist and is therefore not a solution for the differential equation. First expand R^a in powers of r and take the first and second derivative of R^a:

$$R^a = \sum_{k=0}^{\infty} C^a r^k \ , \qquad \partial_r R^a = \sum_{k=0}^{\infty} k C^a r^{k-1} \ , \ \partial_{rr} R^a = \sum_{k=0}^{\infty} (k-1) k C^a r^{k-2} \tag{66}$$

The derivatives of R^a can now be substituted into the differential equation (65). This gives:

$$\begin{aligned} &\lambda \sum_{k=0}^{\infty} C_k^a r^{k+3} - \kappa \sum_{k=0}^{\infty} C_k^a r^{k+1} + \sum_{k=0}^{\infty} C_{k+2}^a (k+1)(k+2) r^{k+3} \\ &+ 3 \sum_{k=0}^{\infty} C_{k+1} (k+1) r^{k+2} + g\epsilon^{abc} u^c \sum_{k=0}^{\infty} C_{k+1}^b (k+1) r^k = 0 \end{aligned} \tag{67}$$

If the assumed power series is a solution the coefficients have to vanish separately for every power in r in the above equation. The coefficients to the individual powers of r in (67) are:

$$\begin{aligned} r^0 &: g\epsilon^{abc} u^c C_1^b = 0 \\ r^1 &: g\epsilon^{abc} u^c 2 C_2^b = \kappa C_0^a \\ r^2 &: g\epsilon^{abc} u^c 3 C_3^b = \kappa C_1^a - 3 C_1^a \\ r^{3+k} &: g\epsilon^{abc} u^c (k+4) C_{k+4}^b = -C_k^a \lambda + C_{k+2}^a \left(\kappa - (k+1)(k+2) - 3(k+2)\right) \end{aligned} \tag{68}$$

Here k is a positive integer or zero. The expression $\kappa - (k+1)(k+2) - 3(k+2)$ is equal to $\kappa - (k+4)(k+2)$ and for the sake of simplicity will from now on be referred to as $b(k)$. The factor $b(k)$ is only zero if $k = l - 2$. For $l = 0, 1$ the factor never vanishes and is negative and in all other cases $b(k)$ is negative for almost all k.
The last equation now reads:

$$r^{3+k} : g\epsilon^{abc} u^c (k+4) C_{k+4}^b = -C_k^a \lambda + C_{k+2}^a b(k) \tag{69}$$

Consider now the system of linear equations of the form $\epsilon^{abc} u^c C^b = w^a$ which needs to be solved for C^a.

$$\begin{aligned} u^3 C^2 - u^2 C^3 &= w^1 \\ u^1 C^3 - u^3 C^1 &= w^2 \\ u^2 C^1 - u^1 C^2 &= w^3 \end{aligned} \tag{70}$$

The coefficient matrix is singular. This system cannot be solved by applying Cramer's rule. In the case of a singular coefficient matrix a linear system of equations has either no solution or infinitely many solutions. This is related to the rank of the coefficient matrix and the rank of the augmented matrix. By performing row reduction a condition for the existence of infinitely many solutions can be obtained:

$$\left(\begin{array}{ccc|c} 0 & u^3 & -u^2 & w^1 \\ -u^3 & 0 & u^1 & w^2 \\ u^2 & -u^1 & 0 & w^3 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & 0 & -u^1/u^3 & -w^2/u^3 \\ 0 & 1 & -u^2/u^3 & w^1/u^3 \\ 0 & 0 & 0 & w^3 + u^2w^2/u^3 + u^1w^1/u^3 \end{array}\right) \tag{71}$$

Using the notation $v^1 = \frac{u^1}{u^3}$ and $v^2 = \frac{u^2}{u^3}$ solutions for this linear system are

$$C^1 = v^1C^3 - \frac{w^2}{u^3} \quad , \quad C^2 = v^2C^3 + \frac{w^1}{u^3} \tag{72}$$

under the condition

$$v^1w^1 + v^2w^2 + w^3 = 0. \tag{73}$$

Using the solutions to this system of equations the recurrence-relations obtained in (68) and (69) can be rewritten:

$$r^0 : \ C_1^1 = v^1C_1^3 \ , C_1^2 = v^2C_1^3 \tag{74}$$

$$r^1 : \ C_2^1 = v^1C_2^3 - \frac{\kappa}{3gu^3}C_0^2 \ , C_2^2 = v^2C_2^3 + \frac{\kappa}{3gu^3}C_0^1 \tag{75}$$

$$r^2 : \ C_3^1 = v^1C_3^3 - \frac{(\kappa-3)}{3gu^3}C_1^2 \ , C_3^2 = v^2C_3^3 + \frac{(\kappa-3)}{3gu^3}C_1^1 \tag{76}$$

$$\begin{aligned} r^{3+k} : \ C_{k+4}^1 &= v^1C_{k+4}^3 - \frac{1}{(k+4)gu^3}\left(-C_k^2\lambda + C_{k+2}^2 b(k)\right) \\ C_{k+4}^2 &= v^1C_{k+4}^3 + \frac{1}{(k+4)gu^3}\left(-C_k^1\lambda + C_{k+2}^1 b(k)\right) \end{aligned} \tag{77}$$

In general there are extra conditions given by (73). Hower for every choice of l and therefore κ one of the following conditions does not apply since in the system of equations the condition (73) is already fulfilled.

$$\kappa\left(v^1C_0^1 + v^2C_0^2 + C_0^3\right) = 0 \tag{78}$$

$$(\kappa-3)\left(v^1C_1^1 + v^2C_1^2 + C_1^3\right) = 0 \tag{79}$$

$$b(k)\left(v^1C_{k+2}^1 + v^2C_{k+2}^2 + C_{k+2}^3\right) = \lambda\left(v^1C_k^1 + v^2C_k^2 + C_k^3\right) \tag{80}$$

For $l = 0$ (78) is always true, for $l = 1$ (79) is always true and for $l = k-2$ the condition (80) is always true. In general there are three coefficients which can be chosen freely. Two of them are coefficients where $k = 0$. The occurrence of the third coefficient depends on the choice of l. One of the coefficients C_k^a where $k = l$ can be chosen freely. This means that for $\lambda \neq 0$ all $v^1C_k^1 + v^2C_k^2 + C_k^3$ vanish for $k < l$. For every other k this sum does not necessarily vanish. It will however be shown that this sum is absolutely convergent for every choice of l.

Consider now only the case of $l = 0$ and therefore $\kappa = 0$. For $l = 0$ the factor $b(k)$ never vanishes and is always negative. First note that the coefficients C_1^a are always vanishing. This can be seen by combining (74) and (79) into

$$\left((v^1)^2 + (v^2)^2 + 1\right) C_3^1 = 0 \tag{81}$$

Therefore C_3^1 vanishes and according to (75) C_1^1 and C_1^2 vanish too. Repeating this process for C_3^a shows that these coefficients are as well equal to zero. Since the equations are recursive and the factor $b(k) \neq 0$ it is possible to combine the conditions and the solutions into new systems of linear equations (84)-(86) which can then be solved uniquely using Cramer's rule.

$$C_2^1 - v^1 C_0^3 = 0 \ , \ C_2^2 - v^2 C_2^3 = 0 \tag{82}$$

$$v^1 C_2^1 + v^2 C_2^2 + C_2^3 = \frac{\lambda}{b(0)} \left(v^1 C_0^1 + v^2 C_0^2 + C_0^3\right) \tag{83}$$

$$C_{k+4}^1 - v^1 C3_{k+4} = -\frac{1}{u^3(k+4)g} \left(b(k) C_{k+2}^2 - \lambda C_k^2\right) \tag{84}$$

$$C_{k+4}^2 - v^2 C3_{k+4} = \frac{1}{u^3(k+4)g} \left(b(k) C_{k+2}^1 - \lambda C_k^1\right) \tag{85}$$

$$v^1 C_{k+4}^1 + v^2 C_{k+4}^2 + C_{k+4}^3 = \frac{\lambda}{b(k+2)} \left(v^1 C_{k+2}^1 + v^2 C_{k+2}^2 + C_{k+2}^3\right) \tag{86}$$

Now certain combinations of coefficients and parameters will be renamed in order to write the solution in a more compact form:

$$U_k^a = b(k) C_{k+2}^a - \lambda C_k^a W_k = v^1 C_k^1 + v^2 C_k^2 + C_k^3 \qquad D = 1 + \left(v^1\right)^2 + \left(v^2\right)^2 \tag{87}$$

The unique solution to the system of linear equations given by (84)-(86) can be determined to be:

$$C_2^1 = \frac{v^1}{D} \frac{\lambda W_0}{b(0)} \ , \ C_2^2 = \frac{v^2}{D} \frac{\lambda W_0}{b(0)} \ , \ C_2^3 = \frac{1}{D} \frac{\lambda W_0}{b(0)} \tag{88}$$

$$C_{k+4}^1 = \frac{1}{D} \left(\frac{\lambda}{b(k+2)} W_{k+2} v^1 - \frac{1}{u^3 g(k+4)} \left(U_k^2 \left(1 + \left(v^2\right)^2\right) + U_k^1 v^1 v^2\right)\right) \tag{89}$$

$$C_{k+4}^2 = \frac{1}{D} \left(\frac{\lambda}{b(k+2)} W_{k+2} v^2 + \frac{1}{u^3 g(k+4)} \left(U_k^1 \left(1 + \left(v^1\right)^2\right) + U_k^2 v^1 v^2\right)\right) \tag{90}$$

$$C_{k+4}^3 = \frac{1}{D} \left(\frac{\lambda}{b(k+2)} W_{k+2} + \frac{1}{u^3 g(k+4)} \left(-v^2 U_k^1 + v^1 U_k^2\right)\right) \tag{91}$$

The coefficients C_0^a can be chosen freely for the case of $l = 0$. The coefficients of a power series where an exponentially decaying factor has been factored out has been derived in appendix B

Now that the recursive expressions for the coefficients of the power series have been

obtained it is necessary to check whether the series converge.
First note that since C_1^a and C_3^a are all equal to zero the coefficients C_n^a for odd n are all vanishing. The formal power series are therefore even. Using the ratio test it can be shown that the Series W_k converges absolutely. All W_k for odd k are zero and the limit of the ratio of two following even coefficients is:

$$\lim_{k\to\infty}\left|\frac{W_{k+2}}{W_k}\right| = \lim_{k\to\infty}\left|\frac{\lambda}{b(k)}\right| = \lim_{k\to\infty}\left|\frac{\lambda}{-(k+2)(k+4)}\right| = 0 < q \leq 1 \tag{92}$$

The series of even W_k is absolutely convergent and therefore the series W_k is absolutely convergent. This is true for all values of l. It has previously been shown that all W_k vanish for $k < l$ and therefore any W_k for $k \geq l$ and k even can be easily written in terms of the l^{th} coefficients as:

$$W_{l+k} = W_l \prod_{\substack{n=l \\ n,k \text{ even}}}^{l+k} \frac{\lambda}{b(n)} = W_l \lambda^{\frac{k}{2}} \prod_{\substack{n=l \\ n,k \text{ even}}}^{l+k} \frac{1}{b(n)} \tag{93}$$

The function $b(k) = l(l+2) - (k+2)(k+4)$ is always negative for $k \geq l$. The series of W_k is therefore alternating for positive eigenvalues λ and not alternating for negative λ.

5. Ideas on showing convergence

In section 4.2 a the coefficients of a formal power series expansion relation for the case $l = 0$ has been derived. In order for them to be a solution to a differential equation the series of the coefficients C_k^a have to converge.

$$W_k = v^1 C_k^1 + v^2 C_k^2 + C_k^3 \tag{94}$$

$$U_k^a = b(k) C_{k+2}^a - \lambda C_k^a \tag{95}$$

$$D = 1 + \left(v^1\right)^2 + \left(v^2\right)^2 \tag{96}$$

$$C_{k+4}^1 = \frac{1}{D}\left(\frac{\lambda}{b(k+2)} W_{k+2} v^1 - \frac{1}{u^3 g(k+4)}\left(U_k^2\left(1 + \left(v^2\right)^2\right) + U_k^1 v^1 v^2\right)\right) \tag{97}$$

$$C_{k+4}^2 = \frac{1}{D}\left(\frac{\lambda}{b(k+2)} W_{k+2} v^2 + \frac{1}{u^3 g(k+4)}\left(U_k^1\left(1 + \left(v^1\right)^2\right) + U_k^2 v^1 v^2\right)\right) \tag{98}$$

$$C_{k+4}^3 = \frac{1}{D}\left(\frac{\lambda}{b(k+2)} W_{k+2} + \frac{1}{u^3 g(k+4)}\left(-v^2 U_k^1 + v^1 U_k^2\right)\right) \tag{99}$$

It has been shown that W_k is absolutely convergent. If the series $v^1 C_k^1$, $v2C_k^2$ and C_k^k have the same sign for almost all k (100) these series all converge since a direct comparison test of each series with W_k would ensure (absolute) convergence.

$$|W_k| = \left|v^1 C_k^1\right| + \left|v^2 C_k^2\right| + \left|C_k^3\right| \tag{100}$$

$$\left|v^1 C_k^1\right| \leq |W_k| \;\; ; \;\; \left|v^2 C_k^2\right| \leq |W_k| \;\; ; \;\; \left|C_k^3\right| \leq |W_k| \tag{101}$$

If $v^1 C_k^1$ converges then C_k^1 will converge as well and the same is true for C_k^2.

$$\sum_{k=0}^{\infty} v^1 C_k^1 = v^1 \sum_{k=0}^{\infty} C_k^1 \tag{102}$$

There are several parameters whose sign can be chosen freely: u^1, u^2,u^3 [5] and the zeroth order coefficients of the formal power series C_0^1, C_0^2 and C_0^3. It has been previously shown that the sign of W_k depends on the coefficients C_0^a and the sign of the eigenvalue λ. For positive λ the sign alternates for negative λ it does not.

6. Summary and outlook

A general expression for the components of a gauge transformed potential in vacuum $A_\mu = i(\partial_\mu h)h^{-1} = \tau^a A_\mu^a$ has been derived:

$$A_\mu^c = -\phi^c \partial_\mu \varphi - (\partial_\mu \phi^c)\sin(\varphi) + (\cos(\varphi) - 1)\,\epsilon^{abc}\phi_\mu^\partial \phi^a$$

Where $h = \exp\left\{i\tau^a \tilde{\phi}^a\right\} = \exp\{i\varphi\tau^a\phi^a\}$, $\varphi = \frac{1}{\sqrt{\tilde{\phi}^a\tilde{\phi}^a}}$ and $\phi^a = \frac{\tilde{\phi}^a}{\varphi}$. A set of transformations $\tilde{\phi}^a = u^a\tilde{\phi}$ has been found that remains in Landau gauge if $\triangle\tilde{\phi} = 0$.
The case of a potential with only a radial component which only depends on the radius in hyper-spherical coordinates $A_\mu^c = A_r^c(r)\boldsymbol{e_r}$ was further examined: It has been shown that only the potential $A_\mu^c = \frac{-u^c}{r^3}\boldsymbol{e_r}$ fulfills the Landau gauge condition.
Using separation of variables for the eigenfunctions $\phi^a(r,\theta,\eta,\varphi) = R^a(r)Y(\theta,\eta,\varphi)$ to eigenvalues λ the eigenproblem of the Fadeev-Popov operator was reduced to a coupled system of ordinary linear differential equations:

$$\left(r^3\lambda - r\kappa\right)R^a + \partial_r\left(r^3\partial_r R^a\right) + g\epsilon^{abc}u^c\partial_r R^b$$

The functions $Y(\theta,\eta,\varphi)$ turn out to be the 4-dimensional spherical harmonics and $-\kappa$ are the eigenvalues of the spherical part of the Laplacian. A power series expansion $R^a(r) = \sum_{k=0}^{\infty} C_k^a r^k$ was used and the coefficients of this formal power series (88) - (91) determined. Convergence or divergence for any of these series was not shown in this thesis but convergence for the combination of these coefficients $W_k = \frac{u^1}{u^3}C_k^1 + \frac{u^2}{u^3}C_k^2 + C_k^3$ was shown and in appendix 5 an idea to show convergence by trying to find an overall sign for the power series was described. The sequence W_k is alternating for positive eigenvalues λ and non-alternating for negative eigenvalues λ.

[5] Remember that $v^1 = \frac{u^1}{u^3}$ and $v^2 = \frac{u^2}{u^3}$

A. Assuming all three eigenfunctions to be equal

Starting with the eigenequations for a potential $A_\mu^c = \frac{u^c}{r^3}\boldsymbol{e_r}$

$$\left(\Delta + \lambda\right)\phi^a + g\epsilon^{abc}A_\mu^c\partial_\mu\phi^b = 0 \tag{103}$$

one might hope to find a solution where all ϕ^b are the same and the system of three differential equations decouples into just one equation. Assuming now that $\phi^b = \phi$ the system of differential equations is now given by:

$$\left(\Delta + \lambda\right)\phi + g\frac{\boldsymbol{e_r}}{r^3}\partial_\mu\phi(u^3 - u^2) = 0 \tag{104}$$
$$\left(\Delta + \lambda\right)\phi + g\frac{\boldsymbol{e_r}}{r^3}\partial_\mu\phi(u^1 - u^3) = 0 \tag{105}$$
$$\left(\Delta + \lambda\right)\phi + g\frac{\boldsymbol{e_r}}{r^3}\partial_\mu\phi(u^2 - u^1) = 0 \tag{106}$$

For the system to decouple into three times the same equation there has to be a solution to the linear system of equations:

$$u^2 - u^3 = y \tag{107}$$
$$u^1 - u^3 = y \tag{108}$$
$$u^2 - u^1 = y \tag{109}$$

The coefficient matrix for this system of linear equations is however singular which means that there are either infinitely many solutions or no solution at all. Considering the augmented matrix one can perform row reduction.

$$\left(\begin{array}{ccc|c} 0 & -1 & 1 & y \\ 1 & 0 & -1 & y \\ -1 & 1 & 0 & y \end{array}\right) \sim \left(\begin{array}{ccc|c} 0 & -1 & 1 & y \\ 1 & 0 & -1 & y \\ 0 & 0 & 0 & 3y \end{array}\right) \tag{110}$$

It can now be seen that this system of linear equations has only infinitely many solutions for y = 0. The resulting system of differential equations in this case is however identical to the case of the untransformed vacuum and has therefore a positive spectrum.

B. Assuming an exponentially decaying factor

One might hope that the system of differential equations

$$\left(r^3\lambda - r\kappa\right) R^a + \partial_r(r^3\partial_r R^a) + g\epsilon^{abc}u^c\partial_r R^b = 0 \tag{111}$$

has a simpler form after factoring out an exponentially decaying factor.

$$R^a(r) = \exp^{-br}\chi^a(r) \tag{112}$$

where b is a positive real number. After taking the derivatives of R^a and substituting them into the system of differential equations the exponential factors cancel and the following new system of differential equations has been obtained:

$$\begin{aligned}\left(r^3\lambda - r\kappa\right)\chi^a + \left(b^2\chi^a - 2b\partial_r\chi^a + \partial_{rr}\chi^a\right) + 3r^2\left(-b\chi^a + \partial_r\chi^a\right)\\ +g\epsilon^{abc}u^c\left(-b\chi^b + \partial_r\chi^b\right) = 0\end{aligned} \tag{113}$$

Now the unknown functions $\chi^a(r)$ are expanded as formal power series:

$$\chi^a = \sum_{k=0}^{\infty} E^a r^k \quad , \qquad \partial_r\chi^a = \sum_{k=0}^{\infty} kE^a r^{k-1} \quad , \quad \partial_{rr}\chi^a = \sum_{k=0}^{\infty}(k-1)kE^a r^{k-2} \tag{114}$$

Substituting these expressions into the differential equation gives:

$$\begin{aligned}\sum_{k=0}^{\infty}\Big(\lambda E_k^a r^{k+3} - \kappa E_k^a r^{k+1} + b^2 E_k^a r^{3+k} - 2b(k+1)E_{k+1}^a r^{3+k} + (k+1)(k+2)E_{k+2}^a r^{k+3}\\ - 3bE_k^a r^{k+2} + 3(k+1)E_{k+1}^a r^{k+2} + g\epsilon^{abc}u^c\left(-bE_k^b r^k + (k+1)E_{k+1}^b r^k\right)\Big) = 0\end{aligned} \tag{115}$$

The right-hand side of the equation is zero, therefore all powers of r need to vanish independently. Thus the following systems of linear equations are obtained:

$$r^0 : g\epsilon^{abc}u^c\left(E_1^b - bE_0^b\right) = 0 \tag{116}$$

$$r^1 : g\epsilon^{abc}u^c\left(2E_2^b - bE_1^b\right) = \kappa E_0^a \tag{117}$$

$$r^2 : g\epsilon^{abc}u^c\left(3E_3^b - bE_2^b\right) = E_1^a(\kappa - 3) + 3bE_0^a \tag{118}$$

$$r^{k+3} : g\epsilon^{abc}u^c\left((k+4)E_{k+4}^b - bE_{k+3}^b\right) = -\left(\lambda + b^2\right)E_k^a + (2k+5)bE_{k+1}^a + b(k)E_{k+2}^a \tag{119}$$

Here k is again any positive integer or zero and $b(k)$ is the same as in the case where no exponential factor has been factored out.

$$b(k) = \kappa - (k+2)(k+4) \tag{120}$$

In section 4.2 it was shown that a system of linear equations of the form $\epsilon^{abc}u^c C^b = w^a$ has infinitely many solutions as long as an extra condition (122) is fullfilled:

$$C^1 = v^1 C^3 - \frac{w^2}{u^3}\ ,\ C^2 = v^2 C^3 + \frac{w^1}{u^3} \tag{121}$$
$$v^1 w^1 + v^2 w^2 + w^3 = 0 \tag{122}$$

The systems (116) - (119) can therefore be rewritten to be in a form uniquely solvable using Cramer's rule:

$$T_k^a = -(\lambda + b^2) E_k^a + E_{k+1}^a (2k+5) b + E_{k+2}^a b(k) \tag{123}$$
$$W_k = v^1 E_k^1 + v^2 E_k^2 + E_k^3 \tag{124}$$

$$\kappa \left(E_0^1 v^1 + E_0^2 v^2 + E_0^3 \right) = 0 \tag{125}$$

$$E_1^1 - v^1 E_1^3 = b \left(E_0^1 - v^1 E_0^3 \right) \tag{126}$$
$$E_1^2 - v^2 E_1^3 = b \left(E_0^2 - v^2 E_0^3 \right) \tag{127}$$
$$(\kappa - 3) \left(E_1^1 v^1 + E_1^2 v^2 + E_1^3 \right) = -3b \left(E_0^1 v^1 + E_0^2 v^2 + E_0^3 \right) \tag{128}$$

$$E_2^1 - v^1 E_2^3 = \frac{b}{2} \left(E_1^1 - v^1 E_1^3 \right) \tag{129}$$
$$E_2^2 - v^2 E_2^3 = \frac{b}{2} \left(E_1^2 - v^2 E_1^3 \right) \tag{130}$$
$$b(0) \left(E_2^1 v^1 + E_2^2 v^2 + E_2^3 \right) = \left(\lambda + b^2 \right) W_0 - 5bW_1 \tag{131}$$

$$E_3^1 - v^1 E_3^3 = \frac{1}{3} \left(b \left(E_2^1 - v^1 E_2^3 \right) - \frac{1}{u^3} \left(E_1^2 (\kappa - 3) + 3bE_0^2 \right) \right) \tag{132}$$
$$E_3^2 - v21 E_3^3 = \frac{1}{3} \left(b \left(E_2^2 - v^2 E_2^3 \right) + \frac{1}{u^3} \left(E_1^1 (\kappa - 3) + 3bE_0^1 \right) \right) \tag{133}$$
$$b(1) \left(E_3^1 v^1 + E_3^2 v^2 + E_3^3 \right) = \left(\lambda + b^2 \right) W_1 - 7bW_2 \tag{134}$$

$$E_{k+4}^1 - v^1 E_{k+4}^3 = \frac{b}{k+4} \left(E_{k+3}^1 - v^1 E_{k+3}^3 \right) - \frac{1}{u^3 (k+4) g} T_k^2 \tag{135}$$
$$E_{k+4}^2 - v^2 E_{k+4}^3 = \frac{b}{k+4} \left(E_{k+3}^2 - v^2 E_{k+3}^3 \right) + \frac{1}{u^3 (k+4) g} T_k^1 \tag{136}$$
$$b(k+2) \left(E_{k+4}^1 v^1 + E_{k+4}^2 v^2 + E_{k+4}^3 \right) = (\lambda + b^2) W_{k+2} - (2k+9) bW_{k+3} \tag{137}$$

There are overall three coefficients that can be chosen freely. One of the equations (125),(128),(131),(134) or (137) is always true dependant on the choice of l. If one of these equations is always true it means that one coefficient of l^{th} order can be chosen freely. The other free coefficients are two of the zeroth coefficients.

References

[1] Axel Maas. Gauge bosons at zero and finite temperature. *Physics Reports 524 (2013), pp. 203-300*, 2011 , arXiv:1106.3942.

[2] IJR Aitchison and AJG Hey. *Gauge theories in particle physics*, volume 2. IOP Publishing, 1989.

[3] J. Zinn-Justin and R. Guida. Gauge invariance. *Scholarpedia*, 3(12):8287, 2008.

[4] Manfred Böhm, Ansgar Denner, and Hans Joos. *Gauge theories of strong and electroweak interactions*. BG Teubner, 2001.

[5] Axel Maas. On the spectrum of the faddeev-popov operator in topological background fields. *Eur.Phys.J.C48:179-192,2006*, 2005 , arXiv:hep-th/0511307.

[6] Christian B Lang and Norbert Pucker. *Mathematische Methoden in der Physik*, volume 2. Springer, 2005.

[7] Jun John Sakurai. *Modern quantum mechanics*. Addison-Wesley, 1994.

[8] Christopher Frye and Costas J. Efthimiou. *Spherical Harmonics in p Dimensions*. arXiv:1205.3548, 2012 , arXiv:1205.3548.