The nucleon as a QCD bound state in a Faddeev approach

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with:

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Motivation

What’s interesting about the nucleon?

- **Description starting from full QCD:**
  Dyson-Schwinger/Bethe-Salpeter/Faddeev approach

- **Guideline:** successful treatment of light mesons.
  
  ... and many more

- **Baryons:** quark-diquark model studies,
  but up to now no fully consistent calculation
  
  ... and many more

**Aim:** ab-initio calculation of the nucleon’s properties
as the simplest exponent of a three-quark bound state in QCD.

⇒ Mass, electromagnetic form factors, magnetic moments, radii.
Motivation

What’s interesting about the nucleon?

- Nucleon = quark core + pion cloud?  
  If yes: what are the ingredients of this quark core?

- Chiral perturbation theory vs. lattice:  
  What can we learn from the  
  Dyson-Schwinger/Bethe-Salpeter/Faddeev approach?

- Proton’s form factor ratio:  
  Rosenbluth separation vs. polarization transfer.  
  Zero crossing?
QCD Bound State Equations

Bound-state poles in 4-point and 6-point functions lead to bound-state equations for mesons and baryons:

\[ P^2 = - M_{\text{Meson}}^2 \]

\[ P^2 = - M_{\text{Baryon}}^2 \]

- Homogeneous integral equations for meson and baryon amplitudes

- Can be solved numerically (iteration) if ingredients are known: Quark propagator, 2- and 3-quark kernel \( K \)
Example: Meson Amplitude

... describes momentum correlation between quark and antiquark:

\[ \Gamma(q, P) = \sum_i f_i (q^2, q \cdot P, P^2) \tau_i(q, P) \]

Coefficients \( f_i \) depend on 3 Lorentz invariants:

- \( q^2, q \cdot P, P^2 = -M^2 \)
- Chebyshev on-shell expansion

Basis elements \( \tau_i(q, P) \) constructed from Clifford algebra:

Pseudoscalar meson: 4 components

\[ \gamma^5 \left\{ I, \not{P}, \not{q}, [\not{P}, \not{q}] \right\} \]

Vector meson: 8 (transverse) components

\[ \gamma^\mu \left\{ I, \not{P}, \not{q}, [\not{P}, \not{q}] \right\} \]

\[ q^\mu \left\{ I, \not{P}, \not{q}, [\not{P}, \not{q}] \right\} \]
Meson BSE

... determines meson bound-state amplitudes $\Gamma$:

\[ \Gamma = K^{(2)} \Gamma \]

But its ingredients:

- the quark propagator
- ... and the 2-quark kernel:

are quark 2- and 4-point functions and must be obtained elsewhere:

from Dyson-Schwinger equations
Infinite coupled system of Dyson-Schwinger equations for QCD’s Green functions:

- Quark propagator:
- Gluon propagator:
- Ghost propagator:
- Ghost-gluon vertex:
- Quark-gluon vertex:

- UV: Perturbation theory
- IR: Infrared exponents
- In between: Truncations necessary!

DSEs

Dyson-Schwinger equation for the quark propagator:

\[
-1 = -1 + \left(1 \text{ Quark propagator:} \right) + \left(2 \text{ Gluon propagator:} \right) + \left(3 \text{ Quark-gluon vertex (Ansatz):} \right)
\]

Quark propagator

\[
S(p)^{-1} = A(p^2) (-i\not{p} + M(p^2)) \quad \text{(dressed)}
\]

\[
S_0(p)^{-1} = Z_2 (-i\not{p} + m_0) \quad \text{(bare)}
\]

2 dressing functions:

- \(M(p^2)\): quark mass function
- \(A(p^2)\): quark renormalization function

Gluon propagator

\[
D_{\mu\nu}(k) = \frac{D(k^2)}{k^2} \left(\delta_{\mu\nu} - \frac{k_\mu k_\nu}{k^2}\right)
\]

Quark-gluon vertex

\[
\Gamma^\mu(p,q) = \Gamma_1(p, q) \gamma^\mu + \ldots
\]

12 tensor components

Ingredients:

- 4 Ghost propagator:
- 3 Quark-gluon vertex (Ansatz):
- 1 Quark propagator:
DSEs

Infinite coupled system of Dyson-Schwinger equations for QCD’s Green functions:


- UV: Perturbation theory
- IR: Infrared exponents
- In between: Truncations necessary!

Quark propagator:

\[ -1 = -1 + + + + \]

Gluon propagator:

\[ -1 = -1 + + + + \]

Ghost propagator:

\[ -1 = -1 + + + \]

Ghost-gluon vertex:

\[ = + + + \]

Quark-gluon vertex:
Quark DSE & Meson BSE

Quark DSE:

\[ D(k^2) = \Gamma(k^2) \gamma^\mu \]

\[ \Gamma(k^2) = \alpha_{\text{eff}}(k^2) + \text{1-Loop} \]

\[ \ln \frac{\pi}{m} \frac{\gamma/m}{k^2 / \Lambda_{\text{QCD}}^2} \]

BSE kernel:

\[ K^{(2)} \]

If chiral symmetry is respected, quark DSE and meson BSE kernel must be related by the AV-WTI

\[ \Rightarrow \text{GMOR:} \ f_\pi m_\pi^2 = 2m_q \langle \bar{q}q \rangle \]

chiral-limit pion is massless

Simplest truncation that preserves chiral symmetry: 

Rainbow-Ladder: retains only vector part \( \sim \gamma^\mu \) of quark-gluon vertex.

\[ \Rightarrow \text{Only one unknown function: effective coupling} \ \alpha_{\text{eff}}(k^2) \]
Use a model coupling for $\alpha_{\text{eff}}(k^2)$, constrained by

- the correct asymptotic behavior
- coupling in the infrared
  strong enough to generate $D\chi_{\text{SB}}$

**Maris/Roberts/Tandy:**


**Quark-mass dependence fixed ...**

- for small quark masses: on observables $m_\pi, f_\pi$
- for large quark masses: on lattice results for the mass function

**Lattice:**


**Similar studies:**

C.S. Fischer and M.R. Pennington, hep-ph/0701123
DSE/BSE approach ...

- clarifies origin of current and constituent quarks; quark condensate (DSE incorporates $D\chi_{SB}$)
- reproduces perturbation theory at large momenta
- is useful for studying confinement (IR behavior of Green functions)
- guarantees chiral symmetry: chiral limit pion is massless (in any truncation that respects AV-WTI)
- Good description of pseudoscalar and vector mesons already in rainbow-ladder truncation


What about baryons?
Baryons

Faddeev Equation:

\[ K^{(3)} = K^{(3)}_{\text{irr}} + K_{1}^{(2)} -1 + K_{2}^{(2)} -1 + K_{3}^{(2)} -1 + \ldots \]

We have already determined the quark propagator.

What’s inside the kernel?

- \( K^{(3)}_{\text{irr}} \)
- \( K_{1}^{(2)} \)
- \( K_{2}^{(2)} \)
- \( K_{3}^{(2)} \)

\( \ldots \)
Baryons

Assumption: 2-quark correlations are the dominant structure in the nucleon
Neglect irreducible 3-quark interactions $K^{(3P)}$: Faddeev Truncation

\[ K_1^{(2)} + K_2^{(2)} + K_3^{(2)} \]

E.g.: Ladder Truncation

But: .... 32 Dirac Tensors -> numerically expensive
Baryons

Assumption: 2-quark correlations are the dominant structure in the nucleon
Neglect irreducible 3-quark interactions $K^{(3P)}$: Faddeev Truncation

\[
\begin{align*}
\phi_i &\equiv K_i^{(2)} + K_2^{(2)} + K_3^{(2)} \\
\end{align*}
\]

E.g.: Ladder Truncation

But: 

\[
\begin{align*}
\phi_i &\equiv \ldots \text{32 Dirac Tensors} \rightarrow \text{numerically expensive}
\end{align*}
\]

Rewrite Faddeev Equation in terms of 2-quark T-Matrix instead of 2-quark kernel:

\[
\begin{align*}
\phi_i &\equiv T_{ik}^{(2)} \phi_k + T_i^{(2)} \phi_j \\
\end{align*}
\]

Now we need an expression for the T-matrix.
The Structure of the T-Matrix

For perturbative momenta, the 2-quark T-Matrix reduces to the ladder kernel:

\[ P \rightarrow \infty \]

In the infrared it may be dominated by a diquark structure:

- Diquark \((\bar{3}) \times \text{quark (3)} = \text{color-singlet nucleon}\)
- Color antisymmetry \(\Rightarrow\) interaction is attractive
- Lattice: strong diquark correlations inside the nucleon

Separable pole ansatz with diquark amplitudes & propagator \((a = \text{scalar, axial-vector, \ldots})\)

\[ T^{(2)} = \sum_a \chi^{(a)} D^{(a)} \bar{\chi}^{(a)} \]

\[ \Rightarrow \text{Colored diquarks as confined constituents of the nucleon} \]
The Structure of the T-Matrix

1) On-shell behavior

If diquark propagator behaves pole-like at some $P^2 = -M^2$, a diquark BSE at that pole (same structure as meson BSE) follows:

Rainbow-Ladder truncation for the sake of consistency:

Pole ansatz in T-matrix is justified if this BSE yields solutions (diquark masses & amplitudes).
The Structure of the T-Matrix

1) On-shell behavior

If diquark propagator behaves pole-like at some $P^2 = -M^2$, a diquark BSE at that pole (same structure as meson BSE) follows:

Rainbow-Ladder truncation for the sake of consistency:

Pole ansatz in T-matrix is justified if this BSE yields solutions (diquark masses & amplitudes).

Rainbow-Ladder does, higher-order additions to the kernel do not (diquark confinement $\sqrt{\ }$).

The Structure of the T-Matrix

Diquark masses:

Mass splitting should decrease with increasing pion mass.

Lattice:
Physical point
\( (m_\pi = 138 \text{ MeV}) \):

- \( m_{sc} = 0.67 \text{ GeV} \)
- \( m_{av} = 0.88 \text{ GeV} \)

Chiral limit:
\( m_{av} - m_{sc} = 0.21 \text{ GeV} \)

Lattice:

\[
m_{av} - m_{sc} = \begin{align*}
0.10(5) \text{ GeV} & \quad \text{I. Wetzorke, F. Karsch: hep-lat/0008008} \\
0.14(1) \text{ GeV} & \quad \text{C. Alexandrou, Ph. de Forcrand, B. Lucini: Phys. Rev. Lett. 97, 222002 (2006)} \\
0.29(4) \text{ GeV} & \quad \text{R. Babich, N. Garron, C. Hoelbling, J. Howard, L. Lellouch, C. Rebbi: hep-lat/0701023} \\
0.36(7) \text{ GeV} & \quad \text{K. Orginos, PoS LAT2005, 054 (2006)}
\end{align*}
\]

Mass splitting should decrease with increasing pion mass.
The Structure of the T-Matrix

Structure of the diquark amplitudes:

\[ \chi(q, P) = \sum_i f_i (q^2, q \cdot P, P^2) \tau_i (q, P) \]

- 4 Dirac components \( \tau_i (q, P) \) for the scalar diquark:
  \[ \gamma^5 C \left\{ I, \not{P}, \not{q}, [\not{P}, \not{q}] \right\} \]

- 8 (transverse) Dirac components \( \tau_i^\mu (q, P) \) for the axial-vector diquark:
  \[ \gamma^\mu C \left\{ I, \not{P}, \not{q}, [\not{P}, \not{q}] \right\} \]
  \[ q^\mu C \left\{ I, \not{P}, \not{q}, [\not{P}, \not{q}] \right\} \]

Depend on 3 complex variables:
- \( q^2 \in \mathbb{C} \): Solve BSE on complex \( q^2 \) plane
- \( q \cdot P \in \mathbb{C} \) via Chebyshev expansion
- \( P^2 \in \mathbb{C} \): Offshell structure?
The Structure of the T-Matrix

If we study composites, we need to know the ingredients at complex arguments.

- BSE and FE sample region inside complex parabolas.
- Singularities in quark propagator and T-Matrix restrict usable domain ⇒ upper limit for masses and photon momenta: \( M_N < M_{N,\text{max}} \); \( Q^2 < Q^2_{\text{max}} \)
- How are the complex values determined?
  - Quark propagator for complex \( q^2 \): DSE
  - Diquark propagator for complex \( P^2 \):
  - Diquark amplitudes for complex \( P^2 \): only mass-shell values are known. Off-shell ansatz necessary!
All in all:

We have:

Faddeev equation:

\[ \phi_i = T_i^{(2)} \phi_j + T_i^{(2)} \phi_k \]

with quark propagator:

\[ \Phi^{-1} = \Phi^{-1} + \Phi^{-1} \]

and 2-quark T-Matrix:

\[ T^{(2)} = \sum \chi D(P^2) \bar{\chi} \]
Combining all of them leads to a **quark-diquark Bethe-Salpeter equation** on the nucleon mass shell:

Binding to the nucleon mediated by **quark exchange** between quark and diquark.

All building blocks determined consistently.
Structure of the quark-diquark amplitudes:

\[ \phi(q, P) = \sum_i Y_i(q^2, \hat{q} \cdot \hat{P}, -M^2) \tau_i(q, P) \Lambda_+(P) \]

Lorentz-invariant coefficients

2 vs. 6 Dirac components \( \tau_i^\mu(q, P) \) for the scalar/axial-vector quark-diquark amplitude:

\[ \begin{align*}
&\{ I, \phi \} \\
&q^\mu\{ I, \phi \}, P^\mu\{ I, \phi \}, \gamma^\mu\{ I, \phi \} 
\end{align*} \]

Correspond in the nucleon’s rest frame to eigenstates of quark-diquark spin and orbital angular momentum

…quark spin

…orbital angular momentum

\( \Rightarrow 3 \ s \ waves, 4 \ p \ waves, 1 \ d \ wave \)
Nucleon mass at physical point:

<table>
<thead>
<tr>
<th>( M_N ) [GeV]</th>
<th>experiment</th>
<th>dominant diquark amplitude</th>
<th>full diquark substructure</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.94</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>0.97</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>0.98</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Correct slope, but 5% smaller than lattice data:

- systematic effect in rainbow-ladder? also \( \rho \) mass 5% below experiment!
- little space for other effects: 3-gluon contributions, pion cloud: at best 100 MeV
Electromagnetic Current

Current conservation requires the following diagrams:


Impulse approximation:

- Photon-quark coupling
- Photon-diquark coupling

Coupling to exchange quark

Vertices depend on dressed propagators

Seagull terms: coupling to diquark amplitudes

Longitudinal parts fixed by Ward-Takahashi identities

Current conservation requires the following diagrams:
Photon-quark coupling

Quark-photon vertex can be written as

\[ \Gamma_{\mu} = \Gamma_{\text{WTI}}^{\mu} + \text{transverse part} \]

- Determined by quark propagator
- Correct asymptotic behavior
- Satisfies WTI

Ball-Chiu Vertex:

\[ \Gamma_{\mu}^{BC}(p, q) = \frac{A(p^2) + A(q^2)}{2} \gamma_{\mu} + \frac{A(p^2) - A(q^2)}{p^2 - q^2} \left( \frac{p+q}{2} \right) (p+q)^{\mu} - i \frac{B(p^2) - B(q^2)}{p^2 - q^2} (p+q)^{\mu} \]

For complete consistency one should solve inhomogeneous BSE for quark photon vertex in Rainbow-Ladder:

\[ (p-q)^{\mu} i \Gamma_{\text{WTI}}^{\mu} = S^{-1}(p) - S^{-1}(q) \]
Construct vertex instead as sum over quark loops:

\[
\Gamma_\mu = \Gamma_{\text{WTI}}^\mu + \text{transverse part}
\]

- Determined by diquark propagator
- Satisfies WTI
- Correct asymptotic behavior

\[
\not p \not q \Gamma \gamma^\mu
\]

Diquark-photon vertex can be written as

\[
\text{Determined by diquark propagator}
\]

\[
\text{Satisfies WTI}
\]

\[
\text{Correct asymptotic behavior}
\]

\[
\text{Sum over Dirac structures; prefactors usually expressed by ansätze: diquark magnetic moment, ...}
\]

- Consistent with diquark propagator (WTI√), no further transverse terms necessary
- Inherits quark-photon vertex
- “Natural” implementation of scalar ↔ axial transition
Electromagnetic Current

Proton’s magnetic moment in Heavy Baryon Chiral Perturbation Theory:


Octet
Decuplet
π,K,η
γ

\[ m^2 (\text{GeV}^2) \]

\[ \mu_p \]

Gernot Eichmann (University of Graz)
Anomalous magnetic moments of proton and neutron

Nearly constant for a wide range of pion masses, finite offset to lattice

At physical point: close to experimental values:

<table>
<thead>
<tr>
<th>n.m.</th>
<th>$\mu_p$</th>
<th>$\mu_n$</th>
</tr>
</thead>
<tbody>
<tr>
<td>exp</td>
<td>2.79</td>
<td>-1.91</td>
</tr>
<tr>
<td>calc</td>
<td>2.80</td>
<td>-1.76</td>
</tr>
</tbody>
</table>

Quark core picture justified!
Electromagnetic Current

Electromagnetic radii of proton and neutron


Main contribution towards chiral limit from pion cloud?

| Physical point: | $r_E^p$ | $r_E^n$ | $r_M^p$ | $r_M^n$ | [fm] |
|----------------|-------|-------|-------|-------|
| exp            | 0.87  | 0.34  | 0.86  | 0.88  |
| calc           | 0.72  | 0.26  | 0.61  | 0.59  |
Electromagnetic Current

Form factors at physical point
Compared to experimental data compiled by P. Grabmayr, Univ. Tübingen (2005)

$Q^2 \,[\text{GeV}^2]$

$G_E^n$

$G_M^n$

$G_E^p$

$G_M^p$
Electromagnetic Current

Form factors at physical point
Compared to experimental data compiled by P. Grabmayr, Univ. Tübingen (2005)

\[ Q^2 \text{[GeV}^2\text{]} \]

\( G_E^p \)

\( G_E^n \)

\( G_M^p \)

\( G_M^n \)

Dominant diquark amplitude only
Electromagnetic Current

Form factors at physical point: dominant contributions

- Dominant contribution: Quark-photon coupling
- All WTIs satisfied ⇒ cancellation of all contributions for $G_E^n (Q^2 \to 0)$
- Diquark-photon vertex: scalar ↔ axial-vector transition only small contribution (in magnetic form factors)
Electromagnetic Current

Proton’s form factor ratio $\mu_p G_E^p / G_M^p$

- Pion cloud contribution only for $Q^2 \lesssim 2 \text{ GeV}^2$:
- Slope at small $Q^2$ due to $r_e > r_m$: pion cloud effect?
- More accurate quark-photon vertex?

From the separable diquark ansatz one cannot extract any information on the off-shell dependence of the diquark amplitudes — nevertheless it is possible to find parameterizations that match the experimental data.

- Pion cloud contribution only for $Q^2 \lesssim 2 \text{ GeV}^2$:
- Slope at small $Q^2$ due to $r_e > r_m$:
- More accurate quark-photon vertex?
- Uncertainties at large $Q^2$ can not be resolved with a separable diquark ansatz alone!

Proton’s form factor ratio $\mu_p G_E^p / G_M^p$

Diquark ansatz cannot reflect full T-Matrix structure:

\[ T^{(2)} = \sum_a \chi^{(a)} D^{(a)} \chi^{(a)} \rightarrow p^2 \rightarrow \infty \]

Timelike diquark poles in the T-Matrix are a rainbow-ladder artifact:

\[ \sum_{\text{I, X, Y, \ldots}} \]

⇒ Determine T-Matrix directly from the kernel, or:

direct solution of truncated Faddeev equation

\[ = \quad + \quad + \]

⇒ would update the status quo on baryons to that of mesons:

• solve a meson BSE in rainbow-ladder
• solve a baryon FE
  (a) without 3-quark interactions
  (b) in rainbow-ladder
Summary

- **Consistent** calculation, mass dependence fixed at quark level.

- The diquark-quark approach well describes the static properties of the nucleon’s **quark core**: mass, magnetic moments, e.m. radii

- Nucleon = quark core + pion cloud holds:
  Pion cloud reduces nucleon mass ($\lesssim 100$ MeV),
  adds to magnetic moments and (materially) to e.m. radii

- Diquark substructure important in $G_E^n$ and $\mu_p G_E^p / G_M^p$

- Form factors at large $Q^2$:
  - off-shell behavior of the T-Matrix (diquark amplitudes)?
  - singularities restrict $Q^2$ range
Next steps

Other observables:

- Nucleon’s pseudoscalar and axial form factors
- Delta and N-Δ transition (Diana Nicmorus)

Technical improvements:

- Inclusion of residue contributions: form factors at large $Q^2$

Structural improvements:

- Quark-photon vertex from inhomogeneous BSE
- Structure of 2-quark T-Matrix
- Direct solution of (truncated) Faddeev equation
- Truncation beyond rainbow-ladder …
Thank you for your attention.