Markus Q. HUBER

PRELIMINARY INFRARED ANALYSIS OF YANG-MILLS GREEN FUNCTIONS IN THE MAXIMALLY ABELIAN GAUGE

June 30th, 2008

The infrared region of Landau gauge Yang-Mills theory has been investigated intensively, especially using the non-perturbative method of Dyson-Schwinger equations. The behavior of vertex functions in the maximally Abelian gauge on the other hand is comparatively unexplored. Expanding the methods known from Landau gauge some general results for the maximally Abelian gauge are deduced and a possible infrared solution in terms of scaling laws is presented.

1 Introduction

One of the hardest problems of modern physics is the description of the strong interaction. The theory of quantumchromodynamics (QCD) proved very successful but is still not completely understood. Especially the problem of confinement, the absence of colored particles from the physical spectrum, eludes the grasp of physicists. Whereas high energy experiments are well described by perturbation theory, we need completely different methods in the low energy regime. During the last ten years functional methods, which are suitable for treating the infrared (IR) region of QCD in a non-perturbative way, were constantly improved and led to new insights. Using among others Dyson-Schwinger equations (DSEs) [1, 2, 3, 4, 5, 6, 7] and renormalization group equations (RGEs) [5, 8] the Landau gauge was investigated extensively. In the following we will generalize some of the employed DSE methods in sec. 2 and apply them to the maximally Abelian gauge (MAG) in sec. 4 to get information about the qualitative behavior of vertex functions in the IR in form of scaling laws. Where useful we will also resort to RGEs. The combination of DSEs and RGEs was first used in [5]. Details about the MAG can be found in sec. 3.

2 Infrared Exponents of Vertex Functions

IR exponents describe how vertex functions behave at low external momenta. Although the loop integrals extend over the complete momentum region, they are dominated by contributions where both the external and the loop momenta are small. For DSEs this is a result of the occurrence of factors like $(p-q)^2$ in the denominator, where p is the external momentum and q the internal one [4]. Taking only the main contributions into account one can replace all quantities by their IR expressions, e.g. power law ansätze for the propagators:

$$D_{\mu\nu}(p^2) = \frac{D(p^2)}{p^2} \xrightarrow{p^2\to 0} A\frac{(p^2)^\delta}{p^2}. \tag{1}$$

Also higher vertex functions can then be described by a combination of powers of the momenta [9]:

$$\begin{aligned}&\Gamma^{\mu_1\cdots\mu_m}(p_1,\cdots,p_n) = \\ &\quad = \sum_t\sum_i c_{i,t}\left(p_1^2/k_i^2,\cdots,p_n^2/k_i^2\right)\left(k_i^2\left(p_1^2,\cdots,p_n^2\right)\right)^{\delta_{i,t}} T_t^{\mu_1\cdots\mu_m}(p_1,\cdots,p_n),\end{aligned} \tag{2}$$

where the $\delta_{i,t}$ are IR exponents for different tensors $T_t^{\mu_1\cdots\mu_m}$ with coefficients $c_{i,t}$. The k_i are suitable combinations of momenta and the sum extends over all possible tensors and a set of independent momenta.

Being only interested in asymptotically small values of the external momenta one can then proceed without having to solve integral equations, but can simply count the exponents of all the momenta using the smallest one for n-point functions to get the qualitative behavior. There are two different cases: When all momenta go to zero identically the counting is straight forward; but if some stay constant while others go to zero more care has to be taken. This case was first investigated in [9] and enlarges the number of IR exponents. Here we are only interested in the former case, which can be generalized to other gauges under certain circumstances. Therefore we have a look at the structure of DSEs and RGEs. For a diagram v appearing in these equations we can write down the IR exponent in a general way using topological relations between the numbers of vertices and propagators:

$$\begin{aligned}
\delta_v = &(\frac{d}{2}-2)(1-\frac{1}{2}\sum_i m^{A_i})-\\
&-\frac{1}{2}\sum_i m^{A_i}\delta_{A_i}+\\
&+\sum_{vertices,k\geq 3} n_d^{A_1\ldots A_k}\left((\frac{d}{4}-1)(k-2)+\delta_{A_1\ldots A_k}+\frac{1}{2}\sum_i k_{A_i}^{A_1\ldots A_k}\delta_{A_i}\right)+\\
&+\sum_{vertices,k\geq 3} n_b^{A_1\ldots A_k}\left((\frac{d}{4}-1)(k-2)+\frac{1}{2}\sum_i k_{A_i}^{A_1\ldots A_k}\delta_{A_i}\right).
\end{aligned} \tag{3}$$

The number of the external propagators of field type A_i is given by m^{A_i} and that of dressed and bare vertices by $n_d^{A_i\ldots A_k}$ and $n_b^{A_i\ldots A_k}$ respectively, where the type of the vertex is described by the upper indices. d is the dimension. The sums extend over all vertices appearing in the diagram. The canonical dimensions were considered explicitly. The $k_{A_i}^{A_1\ldots A_k}$ are defined such that they give the number of times the field A_i appears in the vertex $A_1\ldots A_k$. Let us proceed in four dimensions so that part of the resulting terms vanish. Eq. (3) is valid for integrals of both DSEs and RGEs, but RGEs do not contain bare vertices so the last line can be discarded in this case. Of course there is always the possibility that a diagram vanishes, for instance if the color factor is zero or there are cancelations with other diagrams.

Before we use eq. (3) further, we make a statement about the coefficients of the number of dressed vertices. Using RGEs it can be derived that they are always equal or greater than 0. For three-point functions the argument is that on the right-hand side there is always a diagram that involves three times the three-point function itself. For asymptotically small values of the external momenta we can count the powers of momenta to get the scaling behavior of the diagram. The result gives information about the vertex function in question in form of an inequality for the IR exponents, because the diagram could be the leading one (equality) or a subleading one. E.g. for three different fields A, B and C we get

$$\delta_{ABC}\leq 3\delta_{ABC}+\delta_A+\delta_B+\delta_C \tag{4}$$

from which it can be inferred that

$$\delta_{ABC}+\frac{1}{2}\delta_A+\frac{1}{2}\delta_B+\frac{1}{2}\delta_C\geq 0. \tag{5}$$

This is exactly the coefficient of n_d^{ABC}. For four-point functions one can use the tadpole diagram in the propagator equation,

$$-\delta A \leq \delta_{AABB} + \delta_B \quad \Rightarrow \quad \delta_{AABB} + \delta_A + \delta_B \geq 0, \tag{6}$$

and go for other vertices rescursively:

$$-\delta_A - \delta_B \leq \delta_{AABB} \leq 2\delta_{ABCD} + \delta_C + \delta_D \quad \Rightarrow \quad \delta_{ABCD} + \frac{1}{2}(\delta_A + \delta_B + \delta_C + \delta_D) \geq 0. \tag{7}$$

An extension to higher vertex functions is also possible under certain circumstances. Another useful result are inequalities for the IR exponents of the propagators derived from the coefficients of the $n_b^{A_i...A_k}$. This matter will be discussed in sec. 4 below for the MAG.

A note on the skeleton expansion, which was used in [4] for the purpose of deriving the IR exponents of vertices in the Landau gauge, is in order here: The constraining inequalities derived above correspond exactly to the conditions necessary for the use of the skeleton expansion, i.e. insertions creating higher orders of the expansion have a non-negative IR exponent.

The fact that the coefficients of $n_d^{A_i...A_k}$ are non-negative in eq. (3) shows that in four dimensions the lowest IR exponent possible consists only of the second line, since the third and fourth can only increase the exponent. Of course there may exist further solutions which have different values if the coefficients of $n_d^{A_i...A_k}$ are non-zero, but it is clear that if all δ_{A_i} are 0 there cannot be divergent higher vertex functions.

It is interesting to investigate the consequences of a solution where the above inequalities are saturated. For the RGEs this means that all diagrams have the same IR exponent, which is only determined by the numbers and types of the external legs. As the difference to the corresponding DSE lies in the existence of one bare vertex, but both methods have to give the same exponent, the coefficient of at least one of the $n_b^{A_i...A_k}$ has to be zero. In turn this means that the vertex cannot scale in order to fulfill the assumption of saturated inequalities. In the DSEs then all diagrams involving the same bare vertex have the same IR exponent and the difference of one diagram to the leading one corresponds to the negative IR exponent of its bare vertex. The difference between RGEs and DSEs of having or not having a bare vertex was exploited in [5] to investigate Landau gauge Yang-Mills theory. The case where the inequalities are saturated was also found to be a possible solution for a model of QCD with the quark replaced by a fundamentally charged scalar [10]. The corresponding result for the MAG is presented below.

3 The Maximally Abelian Gauge

This gauge was proposed in the eighties [11, 12, 13] and has some features that make it at first sight very appealing. The underlying idea is to minimize the non-Abelian components of the gauge field in the gauge fixing procedure and thereby favoring Abelian gauge configurations. Lattice calculations [14, 15] implicated that the low energy region of QCD may be described by the Abelian degrees of freedom only, which is called Abelian dominance. The MAG also offers a mechanism for confinement, the dual superconductor [16, 17]. Despite these properties the interest in this gauge was low in recent years. New impulses were given by the work of Schaden [18, 19], who found a non-vanishing ghost condensate leading to a mass of the ghost, which was later on proved to be of tachyonic nature [20, 21]. The behavior of propagators and the existence of the condensate $\langle 1/2 A^a_\mu A^a_\mu + \alpha \bar{c}^a c^a \rangle$ were investigated in [22, 23]. The effects of Gribov copies were taken into account in [24, 25]. The most recent results from ref. [26] indicate finite values for the diagonal gluon, the off-diagonal gluon and the off-diagonal ghost propagators at zero momentum. This picture immediately rises the question where divergent Green functions come from, which could lead to confinement. At the moment it seems nobody can explain confinement with vertex functions that are all regular. Perturbatively the MAG was studied up to three loops in [27].

In this section I would like to present the quantized Lagrangian of Yang-Mills theory in the MAG and shortly comment on the derivation of the DSEs, whereas in the next section I present intermediate results on the behavior of vertex functions and propagators in the MAG employing the results from sec. 2. The first complication of the MAG compared to the Landau gauge is the splitting of the fields, because the diagonal and off-diagonal components of the gluon field are gauge fixed differently. The splitting is defined as

$$A_\mu = A^r_\mu T^r = A^a_\mu T^a + A^i_\mu T^i, \tag{8}$$

where the T^i are the $n-1$ hermitian generators of the SU(N) Cartan subalgebra and the T^a the n^2-n remaining generators. We introduced the very useful notational convention that the indices a, b, ... only count off-diagonal indices and i, j, ... only diagonal ones. Minimizing then the functional

$$R[A] = \int dx A^a_\mu A^a_\mu \tag{9}$$

along the gauge orbits, the gauge fixing condition for the off-diagonal gluon fields turns out to be

$$D_\mu^{ab} A_\mu^b = 0 \tag{10}$$

with the covariant derivative defined with respect to the diagonal gluon components:

$$D_\mu^{ab} := \delta^{ab}\partial_\mu - g\, f^{abi} A_\mu^i. \tag{11}$$

For the diagonal gluons the Landau gauge fixing $\partial_\mu A_\mu^i = 0$ is adopted. Unfortunately gauge fixing is not enough but we also have to add a new term to the Lagrangian for reasons of renormalizability [28, 29]. The quantized version of the Lagrangian is then

$$\mathscr{L} = \mathscr{L}_{YM} + \mathscr{L}_{GF,o} + \mathscr{L}_{GF,d} + \mathscr{L}_R \tag{12}$$

with

$$\mathscr{L}_{YM} = \frac{1}{4} F_{\mu\nu}^a F_{\mu\nu}^a + \frac{1}{4} F_{\mu\nu}^i F_{\mu\nu}^i \tag{13}$$

$$\begin{aligned} \mathscr{L}_{GF,o} = {} & \frac{1}{2\alpha}(D_\mu^{ab} A_\mu^b)^2 + \bar{c}^a D_\mu^{ab} D_\mu^{bc} c^c - \\ & - g\, f^{bcd} \bar{c}^a (D_\mu^{ab} A_\mu^c c^d) - g^2 f^{abi} f^{cdi} A_\mu^b A_\mu^c \bar{c}^a c^d \end{aligned} \tag{14}$$

$$\mathscr{L}_{GF,d} = \frac{1}{2\xi}(\partial_\mu A_\mu^i)^2 \tag{15}$$

$$\begin{aligned} \mathscr{L}_R = {} & \frac{\alpha}{4} g^2 f^{abi} f^{cdi} \bar{c}^a \bar{c}^b c^c c^d - \frac{1}{2} g\, f^{abc} (D_\mu^{ad} A_\mu^d) \bar{c}^b c^c + \\ & + \frac{\alpha}{8} g^2 f^{abc} f^{ade} \bar{c}^b \bar{c}^c c^d c^e - \frac{\alpha}{8} g^2 f^{abe} f^{cde} \bar{c}^a \bar{c}^c c^b c^d. \end{aligned} \tag{16}$$

Eq. (12) was derived from a more general Lagrangian, which can interpolate between Landau gauge and MAG. The diagonal ghosts were integrated out. The remaining parameters ξ and α are set to zero at the end.

To derive the DSEs from this Lagrangian the algorithm described in [30] is very well suited as there is an abundance of interactions. The implementation into a symbolic programming language like *Mathematica* speeds up the process of derivation considerably [30]. For SU(2) (SU(3)) we have 2 (4) three-point interactions and 5 (7) four-point interactions. For simplicity we stick to SU(2) for the remaining part, but all results are equally valid also in SU(3) and the additional interactions do not change the results.

4 Infrared Exponents of Vertex Functions in the Maximally Abelian Gauge

One can get the IR exponents for vertex functions by writing down the system of inequalities and trying to solve it. However, a quicker way for a possible solution is to use eq. (3). In case of the MAG in four dimensions it yields for DSEs

$$\begin{aligned}\delta_v = &-\frac{1}{2}(m^A\delta_A + m^B\delta_B + m^c\delta_c)+\\ &+n_b^{AB^2}(\delta_A/2+\delta_B)+n_d^{AB^2}(\delta_A/2+\delta_B+\delta_{AB^2})+\\ &+n_b^{Ac^2}(\delta_A/2+\delta_c)+n_d^{Ac^2}(\delta_A/2+\delta_c+\delta_{Ac^2})+\\ &+n_b^{A^2B^2}(\delta_A+\delta_B)+n_d^{A^2B^2}(\delta_A+\delta_B+\delta_{A^2B^2})+\\ &+n_b^{A^2c^2}(\delta_A+\delta_c)+n_d^{A^2c^2}(\delta_A+\delta_c+\delta_{A^2c^2})+\\ &+n_b^{B^2c^2}(\delta_B+\delta_c)+n_d^{B^2c^2}(\delta_B+\delta_c+\delta_{B^2c^2})+\\ &+n_b^{B^4}(2\delta_B)+n_d^{B^4}(2\delta_B+\delta_{B^4})+\\ &+n_b^{c^4}(2\delta_c)+n_d^{c^4}(2\delta_c+\delta_{c^4}),\end{aligned} \tag{17}$$

with A the diagonal gluon, B the off-diagonal gluon and c the off-diagonal ghost. Using the inequalities from sec. 2 and the fact that primitively divergent vertices must have an IR exponent lower or equal to 0, we get additional information, e.g.

$$0 \geq \delta_{B^4} \geq -2\delta_B \quad \Rightarrow \quad \delta_B \geq 0. \tag{18}$$

Similarly we get

$$\delta_c \geq 0, \qquad \delta_A+\delta_B \geq 0, \qquad \delta_A+\delta_c \geq 0. \tag{19}$$

From this we see that the off-diagonal gluon and ghost propagators cannot be IR enhanced. As mentioned above such conclusions can always be nullified by explicit cancelations or vanishing diagrams, so that a further investigation of this important result is necessary. Using for the proof explicit diagrams of three-point DSEs, the same argument as in [9] applies: A cancelation is very unlikely except diagrams with the same kinematic dependence exist, which means in this case that the diagram can only depend on one instead on three external momenta. In Landau gauge no such diagrams occur in the gluon propagator DSE, but in MAG there are also other diagrams of this structure.[1] Calculating only the color factors of the diagrams, thereby assuming the tree-level factor for the dressed three-point functions, reveals that the possibly dangerous diagrams are zero. Therefore they cannot invalidate the above argument, but show that one always has to consider the possibility of vanishing diagrams. However, if the diagram used for derivation of an inequality vanishes, it is in many cases still possible to obtain the same inequality from another DSE/RGE.

To get a first possible solution for the vertices, we assume that the inequalities from the RGE are saturated. The leading diagrams in the DSE then turn out to be those with a bare *AAcc* or *AABB* interaction. This is a rather unexpected result, since two-loop diagrams are the leading ones. The IR exponent of the diagonal gluon is then the negative of those of the other two propagators, which are equal. This solution can furthermore be only valid, if one renormalizes the diagonal gluon DSE such that the bare propagator at zero momentum vanishes. This is a general statement for IR enhanced propagators: They can only diverge for small momenta, if the zero momentum bare propagator drops out in the renormalization process. As an example, where this is indeed the case, may again serve Landau gauge Yang-Mills theory. The IR exponent for a MAG vertex with m diagonal gluons, n off-diagonal gluons and o off-diagonal ghosts is

$$\delta_{m,n,o} = \frac{1}{2}(m-n-o)\kappa, \tag{20}$$

where κ is an unknown positive constant identified as the IR exponent of the off-diagonal propagators.

5 Conclusions and Outlook

In this paper we derived a possible IR scaling solution for the MAG. Thereby a general method of deriving the IR exponent of a vertex function was employed. It was shown that the off-diagonal propagators cannot be enhanced in the IR and the solution presented features a diagonal gluon propagator with the negative IR exponent of the diagonal propagators. The results can be expressed using one unknown constant and all vertices are IR enhanced or do not scale.

The next step in this work will be the search for other solutions. Those necessarily have greater

[1] The diagram is the so-called swordfish diagram with a dressed three-point and a bare four-point vertex. The latter is responsible for the dependence on only one external momentum.

IR exponents as the proposed solution is the only one that fulfills the constraints for a maximal divergent solution. Explicit calculations of some diagrams may give a numeric value for the unknown constant. Also the necessary condition for the bare diagonal gluon propagator in the renormalization process deserves deeper investigation.

Acknowlegdments

I would like to thank my supervisor Reinhard Alkofer for his support of my work. Thanks also to Kai Schwenzer, with whom I enjoyed many discussions. I am grateful to Christian S. Fischer and Jan Pawlowski for telling me more about scaling and the use of DSEs and RGEs, and to David Dudal and Silvio P. Sorella for discussions about the MAG.

References

[1] L. von Smekal, A. Hauck, and R. Alkofer, *Ann. Phys.* **267** (1998) 1, `hep-ph/9707327`.

[2] R. Alkofer and L. von Smekal, *Phys. Rept.* **353** (2001) 281, `hep-ph/0007355`.

[3] C. Lerche and L. von Smekal, *Phys. Rev.* **D65** (2002) 125006, `hep-ph/0202194`.

[4] R. Alkofer, C. S. Fischer, and F. J. Llanes-Estrada, *Phys. Lett.* **B611** (2005) 279–288, `hep-th/0412330`.

[5] C. S. Fischer and J. M. Pawlowski, *Phys. Rev.* **D75** (2007) 025012, `hep-th/0609009`.

[6] C. S. Fischer, *J. Phys.* **G32** (2006) R253–R291, `hep-ph/0605173`.

[7] R. Alkofer, C. S. Fischer, F. J. Llanes-Estrada, and K. Schwenzer, *Ann. Phys., in print* (2008) , `arXiv:0804.3042 [hep-ph]`. Ann. Phys., in print.

[8] J. M. Pawlowski, D. F. Litim, S. Nedelko, and L. von Smekal, *Phys. Rev. Lett.* **93** (2004) 152002, `hep-th/0312324`.

[9] R. Alkofer, M. Q. Huber, and K. Schwenzer, `arXiv:0801.2762 [hep-th]`.

[10] L. Fister, “diploma thesis in preparation.” diploma thesis in preparation, 2008.

[11] G. ’t Hooft, *Nucl. Phys.* **B190** (1981) 455.

[12] A. S. Kronfeld, G. Schierholz, and U. J. Wiese, *Nucl. Phys.* **B293** (1987) 461.

[13] A. S. Kronfeld, M. L. Laursen, G. Schierholz, and U. J. Wiese, *Phys. Lett.* **B198** (1987) 516.

[14] T. Suzuki and I. Yotsuyanagi, *Phys. Rev.* **D42** (1990) 4257–4260.

[15] S. Hioki *et al.*, *Phys. Lett.* **B272** (1991) 326–332.

[16] Y. Nambu, *Phys. Rev.* **D10** (1974) 4262.

[17] S. Mandelstam, *Phys. Rept.* **23** (1976) 245–249.

[18] M. Schaden, `hep-th/9909011`.

[19] K.-I. Kondo and T. Shinohara, *Phys. Lett.* **B491** (2000) 263–274, `hep-th/0004158`.

[20] D. Dudal and H. Verschelde, *J. Phys.* **A36** (2003) 8507–8516, `hep-th/0209025`.

[21] H. Sawayanagi, *Phys. Rev.* **D67** (2003) 045002.

[22] K.-I. Kondo, *Phys. Lett.* **B514** (2001) 335–345, `hep-th/0105299`.

[23] D. Dudal *et al.*, *Phys. Rev.* **D70** (2004) 114038, `hep-th/0406132`.

[24] M. A. L. Capri, V. E. R. Lemes, R. F. Sobreiro, S. P. Sorella, and R. Thibes, *Phys. Rev.* **D72** (2005) 085021, `hep-th/0507052`.

[25] M. A. L. Capri, V. E. R. Lemes, R. F. Sobreiro, S. P. Sorella, and R. Thibes, *Phys. Rev.* **D74** (2006) 105007, `hep-th/0609212`.

[26] M. A. L. Capri, V. E. R. Lemes, R. F. Sobreiro, S. P. Sorella, and R. Thibes, *Phys. Rev.* **D77** (2008) 105023, `arXiv:0801.0566 [hep-th]`.

[27] J. A. Gracey, *JHEP* **04** (2005) 012, `hep-th/0504051`.

[28] H. Min, T. Lee, and P. Y. Pac, *Phys. Rev.* **D32** (1985) 440.

[29] A. R. Fazio, V. E. R. Lemes, M. S. Sarandy, and S. P. Sorella, *Phys. Rev.* **D64** (2001) 085003, `hep-th/0105060`.

[30] R. Alkofer, M. Q. Huber, and K. Schwenzer, `0808.2939 [hep-th]`.