Derivation of Dyson-Schwinger equations

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Dyson-Schwinger equations, the equations of motion of Green functions, provide a non-perturbative description of quantum field theories. Especially for the strong interaction they have been used very successfully in recent years. In this note their derivation is described. As examples the two-point Dyson-Schwinger equation for a scalar theory and the ghost-gluon vertex Dyson-Schwinger equations of in Yang-Mills theory are given.
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I. INTRODUCTION

Dyson-Schwinger equations (DSEs) are named after F. J. Dyson [1] and J. S. Schwinger [2, 3]. Put into one sentence, DSEs are the equations of motion of Green functions and describe the propagation and interaction of the fields of a theory. The full system of DSEs provides a complete description of the theory1. Therefore DSEs can be used to generate a perturbative expansion in the weak coupling regime, but they show their true strength when applied for strong coupling. This makes them a perfect tool for the investigation of Yang-Mills theory and also quantum chromodynamics (QCD), where non-perturbative methods are needed to explain such phenomena as confinement and dynamical chiral symmetry breaking. Both are important properties of these theories and cannot be accounted for by standard perturbation theory.

DSEs have the following advantages:

• They are a continuum method, i.e., different scales are relatively easily accessible. Both the UV and the IR regions can be investigated. On the lattice the accessibility of these regions is constrained by the lattice spacing and size.
• At finite density DSEs do not have a sign problem as on the lattice.
• The chiral limit is easily accessible.
• The equations are exact.

These traits make DSEs seem like the perfect tool. Unfortunately, there is one big drawback: DSEs form an infinite tower of coupled integral equations. Therefore the system has to be truncated and cannot be solved completely. However, nowadays much is known about constraints on these truncations and there are some good guidelines to follow. There are even cases, where the complete tower can be solved qualitatively in an asymptotic regime. Before I explain how the infinite tower is built, I will describe the content of a DSE.

The building blocks of DSEs are the Green functions of the theory, which describe how a particle/field propagates and interacts. They are also called correlation functions or n-point functions. One can distinguish propagators, which are the inverse of the two-point functions, and vertices, which are n-point functions with n > 2. Thus an intuitive representation is provided by diagrams. In complete analogy to the diagrams one knows from perturbation theory a line is a propagator and a crossing of lines a vertex, but in contrast to perturbation theory here we have non-perturbatively exact quantities. They are denoted by an additional blob. The conventional symbols are depicted in table 1. However, in order not to clutter the diagrams one often refrains from drawing the blobs of the internal propagators. Also in this note all internal propagators are to be taken as full propagators.

For a scalar field theory with cubic and quartic interactions the DSE of the two-point function is depicted in fig 1. This DSE can be rewritten such that we have the DSE of the propagator, but we stick here and in the following with the two-point equations as they follow directly from the derivation and we gain no new insights by rewriting. Intuitively this picture can be understood as follows: On the left-hand side we have the full two-point function/inverse

1Note that the solution of the equations might require a sort of boundary condition which is required as external input. A prominent example is the ghost-gluon system of Landau gauge Yang-Mills theory, where the value of the ghost dressing function at zero momentum can be chosen as such a condition.
propagator. Thus on the right-hand side there are all possible ways of propagation for a scalar field. First, it can just propagate without any interaction whatsoever. This is represented by the bare two-point function. Then it can split into two other fields, which propagate in all possible ways themselves and recombine again in all possible ways to one field. Note that the splitting into two fields happens via the bare vertex. This is a trait of all DSEs: Every diagram on the right-hand side contains exactly one bare n-point function. In case of the splitting via the four-point function, there are three ways of how the fields can propagate and recombine. The numerical factors are similar to perturbation theory symmetry factors derived from the number of possible permutations that leave the diagram unchanged. Sometimes they are not written explicitly. The signs in the graphical representation depend on the employed conventions and vary from source to source. All DSEs follow a similar pattern and understanding it allows a good guess at the basic form of DSEs only from the type of interactions.

The two-point DSE contains dressed three- and four-point functions. Consequently those functions are required, if we want to solve this DSE. As their DSEs contain dressed five- and six-point functions, the system of DSEs can never be closed and leads to the infinite tower of DSEs. For an explicit solution one has to truncate a DSE: Either one neglects certain Green functions or one makes an ansatz for them. Ideally this ansatz is constrained by further knowledge about these functions.

A similar hierarchy of functional equations is given by exact renormalization group equations (ERGEs). The most striking differences are that they consist only of one-loop diagrams, have no bare quantities and depend on an artificial momentum scale that interpolates between the classical and the full effective action. A short comparison between the two sets of functional equations is given in tab. II.

In the following the derivation of DSEs is first explained for a scalar theory and then generalized to other theories. Finally the ghost-gluon vertex DSEs of Landau gauge Yang-Mills theory are derived.

**TABLE I:** Graphical representation of Green functions.

<table>
<thead>
<tr>
<th>DSEs</th>
<th>ERGEs</th>
</tr>
</thead>
<tbody>
<tr>
<td>one bare quantity per diagram</td>
<td>only dressed quantities</td>
</tr>
<tr>
<td>one-loop and more (depends on interactions)</td>
<td>one-loop only</td>
</tr>
<tr>
<td>effective action</td>
<td>effective average action</td>
</tr>
<tr>
<td>exact equations</td>
<td>exact equations</td>
</tr>
<tr>
<td>(but truncations required)</td>
<td>(but truncations required)</td>
</tr>
</tbody>
</table>

**TABLE II:** Comparison of DSEs and ERGEs.

![FIG. 1: DSE of the scalar two-point function. Some diagrams on the right-hand side have names: The last three diagrams are the tadpole, the sunset and the squint.](image-url)
II. DERIVATION OF DYSON-SCHWINGER EQUATIONS

There are various ways of deriving DSEs. The preferred one is via the path integral. A derivation with the canonical formalism is possible, but less instructive and definitely more tedious. The interested reader is referred to ref. [4]. The derivation is first demonstrated for a simple scalar field theory. Based on this the generalization to other theories is discussed.

A. Scalar field theory

To start I recapitulate some basic quantum field theory by introducing the generating functionals of a scalar theory. From the action

\[ S[\phi] = \int dx \left( \phi(-\partial^2 + m^2)\phi + \frac{\lambda_3}{3!}\phi^3 + \frac{\lambda_4}{4!}\phi^4 \right) \]  

(1)

we construct the path integral

\[ Z[J] = \int D[\phi] e^{-S[\phi] + \int dx \phi(x)J(x)} = e^{W[J]}, \]  

(2)

where \( J(x) \) is the source of the field. The path integral \( Z[J] \) is also called the generating functional for full Green functions and \( W[J] \) that for connected Green functions, i.e., Green functions which correspond to only one connected diagram. A Legendre transform of \( W[J] \) with respect to the source yields the so-called effective action that generates the one-particle irreducible (1PI) Green functions\(^2\), which are those Green functions that are still connected after one internal line is cut:

\[ \Gamma[\Phi] = -W[J] + \int dx \Phi(x)J(x). \]  

(3)

It depends on the averaged field \( \Phi \) in the presence of the external current \( J \),

\[ \Phi(x) := \langle \phi(x) \rangle_J = \frac{\delta W[J]}{\delta J(x)} = Z[J]^{-1} \int D[\phi]\phi(x)e^{-S[\phi] + \int dx \phi(y)J(y)}. \]  

(4)

The current, on the other hand, can be expressed as the derivative of the effective action:

\[ J(x) = \frac{\delta \Gamma[\Phi]}{\delta \Phi(x)}. \]  

(5)

The n-point functions are computed from the n-th derivative of the effective action. For vertices (\( n > 2 \)) we use the convention that the vertex is the negative of the derivative. This will avoid cumbersome sign changes in the DSEs and one can directly determine the correct sign:

\[ \Gamma(x_1, \ldots, x_n) := \frac{\delta \Gamma[\Phi]}{\delta \Phi(x_1) \cdots \delta \Phi(x_n)}, \quad n > 2. \]  

(6)

The \( \Gamma(x_1, \ldots, x_n) \) are not yet the physical n-point functions of the theory because the external sources \( J \) are still non-vanishing as indicated by the superscript \( J \). Setting them to zero physical propagators \( D(x - y) \) and vertices \( \Gamma(x_1, \ldots, x_n) \) are obtained\(^3\):

\[ D(x - y) := D(x, y)^{J=0}, \]  

(7)

\[ \Gamma(x_1, \ldots, x_n) := \Gamma(x_1, \ldots, x_n)^{J=0}. \]  

(8)

---

\(^2\) 1PI functions are sufficient for a complete description, as one can construct all connected and full Green functions from them.

\(^3\) Due to translation invariance the physical propagator depends only on the difference of two points.
The function \( D(x,y) \), being the inverse of the two-point function\(^4\), is given by
\[
D(x,y) := \frac{\delta^2 W[J]}{\delta J(x) \delta J(y)} = \left( \frac{\delta^2 \Gamma[\Phi]}{\delta \Phi(x) \delta \Phi(y)} \right)^{-1}.
\]

It is important in the derivation of DSEs that the sources \( J \) are set to zero only at the end. Otherwise one could miss contributions.

For the derivation of DSEs we start with the integral of a total derivative, which vanishes:
\[
0 = \int D[\phi] \frac{\delta}{\delta \phi} e^{-S + \int d\phi(y) J(y)} = \int D[\phi] \left( - \frac{\delta S}{\delta \phi(x)} + J(x) \right) e^{-S + \int d\phi(y) J(y)} = \left( - \frac{\delta S}{\delta \phi(x)} \right)_{\phi(x')=\phi(x')} + J(x) Z[J] = 0.
\]

Here the derivative of the action and the source are pulled out of the path integral by replacing the fields with derivatives with respect to the sources. Employing further derivatives with respect to sources yields the DSEs of the full Green functions, but our goal are the DSEs for the 1PI functions. So we continue by substituting \( Z[J] \) with \( e^{W[J]} \) and using
\[
e^{-W[J]} \left( \frac{\delta}{\delta J(x)} \right) e^{W[J]} = \frac{\delta W[J]}{\delta J(x)} + \frac{\delta}{\delta J(x)}.
\]

After multiplication of eq. \((10)\) from the left with \( e^{-W[J]} \) we find
\[
- \frac{\delta S}{\delta \phi(x)} \bigg|_{\phi(x')=\phi(x')} + J(x) = 0.
\]

This is the generating DSE for connected correlation functions. The DSEs of connected Green functions are obtained by acting with further source derivatives on eq. \((12)\). To get the corresponding version for 1PI functions we perform the Legendre transformation of \( W[J] \) with respect to the source \( J \). Thereby \( \delta W[J]/\delta J(x) \) changes to \( \Phi(x) \) and \( \delta/\delta J(x) \) becomes
\[
\frac{\delta}{\delta J(x)} = \int dz \frac{\delta \Phi(z)}{\delta J(x)} \frac{\delta}{\delta \Phi(z)} = \int dz \frac{\delta W[J]}{\delta J(x)} \frac{\delta}{\delta \Phi(z)} = \int dz \frac{\delta^2 W[J]}{\delta J(x) \delta J(z)} \frac{\delta}{\delta \Phi(z)} = \int dz D(x,z) \frac{\delta}{\delta \Phi(z)}.
\]

This yields
\[
- \frac{\delta S}{\delta \phi(x)} \bigg|_{\phi(x')=\Phi(x')+\int dz D(x',z) \delta/\delta \Phi(z)} + \frac{\delta \Gamma[\Phi]}{\delta \Phi(x)} = 0,
\]
which is the basic equation. All DSEs for 1PI Green functions can be derived from it by further differentiations with respect to the fields.

Let us work out the first expression. For this it is advantageous to express the action eq. \((1)\) as\(^5\)
\[
S[\phi] = \frac{1}{2} \int dxdy \phi(x) S^{(2)}(x,y) \phi(y) - \frac{1}{4!} \int dxdydz S^{(3)}(x,y,z) \phi(x) \phi(y) \phi(z) - \frac{1}{4!} \int dxdydzdu S^{(4)}(x,y,z,u) \phi(x) \phi(y) \phi(z) \phi(u).
\]

\(^4\) It should be stressed that for a theory with more than one field this is a matrix relation, i.e., if the two-point matrix is not diagonal there is a non-trivial relationship between propagators and two-point functions. This complicates analyses of actions with mixed propagators like the Gribov-Zwanziger action. For a diagonal matrix in case of bosons or an off-diagonal matrix for fermions the situation is simpler, because the propagator can be directly calculated as the inverse of the corresponding two-point function.

\(^5\) Using the explicit expressions for the action may look simpler at this point, but at the end it is sometimes hard to identify the bare vertices correctly. As an example one can think of the bare gluonic vertices of QCD, which are more complicated than those of a scalar theory because of color and Lorentz structures.
The derivatives act on field and propagators as follows:

\[
S^{(2)}(x, y) = \left. \frac{\delta^2 S}{\delta \phi(x) \delta \phi(y)} \right|_{\phi=0} = \delta(x - y)(-\partial^2 + m^2)
\]

(16)

and the bare vertices by

\[
S^{(3)}(x, y, z) = -\left. \frac{\delta^3 S}{\delta \phi(x) \delta \phi(y) \delta \phi(z)} \right|_{\phi=0} = \lambda_3 \delta(x - y) \delta(x - z),
\]

(17)

\[
S^{(4)}(x, y, z, u) = -\left. \frac{\delta^4 S}{\delta \phi(x) \delta \phi(y) \delta \phi(z) \delta \phi(u)} \right|_{\phi=0} = \lambda_4 \delta(x - y) \delta(x - z) \delta(x - u).
\]

(18)

The first term in eq. (14) then becomes

\[
\left. \frac{\delta S}{\delta \phi(x)} \right|_{\phi(x') = \Phi(x') + \int dzD(x', z)^J \delta/\delta \Phi(z)} = \int du S^{(2)}(x, u) \Phi(u) - \int du dv \frac{1}{2} S^{(3)}(x, u, v) \Phi(u) \Phi(v) - \frac{1}{3!} \int du dv dw S^{(4)}(x, u, v, w) \Phi(u) \Phi(v) \Phi(w) \bigg|_{\phi(x') = \Phi(x') + \int dzD(x', z)^J \delta/\delta \Phi(z)}
\]

(19)

\[
= \int du S^{(2)}(x, u) \Phi(u) - \frac{1}{2} \int du dv S^{(3)}(x, u, v) \left( \Phi(u) + \int dzD(u, z)^J \delta/\delta \Phi(z) \right) \Phi(v) - \frac{1}{3!} \int du dv dw S^{(4)}(x, u, v, w) \left( \Phi(u) + \int dzD(u, z)^J \delta/\delta \Phi(z) \right) \left( \Phi(v) + \int dyD(v, y)^J \delta/\delta \Phi(y) \right) \Phi(w).
\]

The derivatives act on field and propagators as follows:

\[
\frac{\delta}{\delta \Phi(y)} \Phi(x) = \delta(x - y),
\]

(20a)

\[
\frac{\delta}{\delta \Phi(x)} D(y, z)^J = \frac{\delta}{\delta \Phi(x)} \left( \frac{\delta^2 \Gamma[\Phi]}{\delta \Phi(y) \delta \Phi(z)} \right)^{-1} = -\int dz_1 dz_2 D(y, z_1)^J \frac{\delta^2 \Gamma[\Phi]}{\delta \Phi(y) \delta \Phi(z_1)} \left( \frac{\delta^2 \Gamma[\Phi]}{\delta \Phi(z_1) \delta \Phi(x) \delta \Phi(z_2)} \right) \left( \frac{\delta^2 \Gamma[\Phi]}{\delta \Phi(z_2) \delta \Phi(z)} \right)^{-1} = \int dz_1 dz_2 D(y, z_1)^J \Gamma(z_1, x, z_2)^J D(z_2, z)^J,
\]

(20b)

\[
\frac{\delta}{\delta \Phi(x)} \Gamma(y_1, \ldots, y_n)^J = -\frac{\delta \Gamma[\Phi]}{\delta \Phi(x) \delta \Phi(y_1) \cdots \delta \Phi(y_n)} = \Gamma(x, y_1, \ldots, y_n)^J.
\]

(20c)

For completeness the derivative of an n-point function was included. The second relation follows from the matrix relation

\[
\delta(M M^{-1}) = 0 \quad \Rightarrow \quad \delta M^{-1} = -M^{-1}(\delta M) M^{-1}.
\]

(21)

We can use eqs. (20) in eq. (19) and get

\[
\left. \frac{\delta S}{\delta \phi(x)} \right|_{\phi(x') = \Phi(x') + \int dzD(x', z)^J \delta/\delta \Phi(z)}
\]
\[ = \int duS^{(2)}(x, u)\Phi(u) - \frac{1}{2} \int dudsS^{(3)}(x, u, v) \left( \Phi(u)\Phi(v) + D(u, v)^{I} \right) - \frac{1}{3!} \int dudwdS^{(4)}(x, u, v, w) \left( \Phi(u)\Phi(v)\Phi(w) + 3\Phi(u)D(v, w)^{I} + \right. \\
+ \int dzD(u, z)^{I} \int dv_{1}dv_{2}D(v, v_{1})^{I}\Gamma(v_{1}, z, v_{2})^{I}D(v_{2}, w)^{I} \right), \] (22)

where the symmetry of the \( S^{(1)} \) under exchange of their arguments was used.

Now we can derive the DSE for the two-point function by plugging this into eq. (14) and differentiating with respect to \( \Phi(y) \):

\[ \Gamma(x, y) = S^{(2)}(x, y) - \int duS^{(3)}(x, u, y)\Phi(u) - \frac{1}{2} \int dudsS^{(3)}(x, u, v) \int dz_{1}dz_{2}D(u, z_{1})^{I}\Gamma(z_{1}, y, z_{2})^{I}D(z_{2}, v)^{I} - \frac{1}{2} \int dudsS^{(4)}(x, y, u, v)\Phi(u)\Phi(v) - \frac{1}{2} \int dudsS^{(4)}(x, y, u, v)D(u, v)^{I} - \frac{1}{2} \int dudwdS^{(4)}(x, u, v, w)\Phi(u) \int dv_{1}dv_{2}D(v, v_{1})^{I}\Gamma(v_{1}, z, v_{2})^{I}D(v_{2}, w)^{I} - \frac{1}{2} \int dudwdS^{(4)}(x, y, u, v) \int dzD(u, z)^{I} \int dv_{1}dv_{2}D(v, v_{1})^{I}\Gamma(y, v_{1}, z, v_{2})^{I}D(v_{2}, w)^{I} - \frac{1}{2} \int dudwdS^{(4)}(x, u, v, w) \int dzdz_{1}dz_{2}D(u, z_{1})^{I}\Gamma(z_{1}, y, z_{2})^{I}D(z_{2}, z)^{I} \times \int dv_{1}dv_{2}D(v, v_{1})^{I}\Gamma(v_{1}, z, v_{2})^{I}D(v_{2}, w)^{I}. \] (23)

We can set the sources to zero and the first terms in the second and third lines vanish. The last step is the Fourier transformation to momentum space:

\[ \int dxdye^{i(p-x)\cdot y}\Gamma(x, y) = \int dxdye^{i(p-x)\cdot y} \int drds e^{-i(r+x+sy)}\Gamma(r, s) = \int drds \delta(p-r)\delta(p+s)\Gamma(r, s) = \]

\[ = \Gamma(p, -p) = S^{(2)}(p, -p) - \int dqS^{(3)}(p, p + q, -q)D(p + q)\Gamma(-p, q, -p - q)D(q) - \frac{1}{2} \int dqS^{(4)}(p, -p, q, -q)D(q) - \frac{1}{3!} \int dqrdS^{(4)}(p, q, r, -p - q - r)D(q)D(r)\Gamma(-p, -q, -r, p + q + r)D(-p - q - r) - \frac{1}{2} \int dqrdS^{(4)}(p, -p - q, r + q, -r)D(p + q)\Gamma(-r, -q, r)D(r)\Gamma(-p, p + q, -q)D(q)D(r + q), \] (24)

where

\[ dp := \frac{dp}{(2\pi)^{d}}, \] (25)

\[ \delta(p) := (2\pi)^{d}\delta(p). \] (26)

The graphical representation can be found in fig. 1.

DSEs for higher n-point functions can be derived by applying further derivatives to eq. (23), setting the sources to zero and doing the Fourier transformation.

The case of broken symmetry, when the vacuum expectation value of \( \phi \) is not 0, can also be treated. Then setting the field values to the physical expectation value, say, \( \langle \phi \rangle_{J=0} = \Phi \), yields an infinite series for the n-point functions and external fields in eq. (23) can survive:

\[ \frac{\delta\Gamma[\Phi]}{\delta\Phi(x_{1}) \cdots \delta\Phi(x_{n})} \bigg|_{J=0} \]
= \Gamma(x_1, \ldots, x_n)^J=0 + \int dy \Gamma(x_1, \ldots, x_n, y)^J=0 \Phi(y) + \frac{1}{2} \int dy dz \Gamma(x_1, \ldots, x_n, y, z)^J=0 \Phi(y) \Phi(z) + \ldots. \quad (27)

B. Generalization to other theories

Even for widely used actions like Yang-Mills theory in the Landau gauge the derivation of DSEs is quite cumbersome. One reason is simply the growth of the number of diagrams when one proceeds to higher vertex functions. Another one is the complications introduced when several different fields are present, because then the n-point functions are matrices. A convenient way to deal with multiple fields is by using a superfield \( \phi_i \) with one index, which represents all indices, the space/momentum dependence and the field type. In the scalar theory this field stands for \( \phi(x) \), whereas in Landau gauge QCD it is the set of gluon, ghost and quark fields: \( \{ A^a_\mu(x), c^i(x), c^i(x), q^i(x), \bar{q}^j(x) \} \). Repeated indices are summed and integrated over as appropriate, i.e., this includes also a sum over all fields.

The usefulness of this notation becomes apparent in the relation between the propagators and two-point functions for non-vanishing external sources (\( \Phi_i \)):

\[ \delta_{ij} = \frac{\delta \Phi_i}{\delta \Phi_j} = \delta J_k \frac{\delta \Phi_i}{\delta \Phi_k} \frac{\delta W[J]}{\delta J_k}. \quad (28) \]

The index \( k \) includes a summation over all fields. The importance of this can be seen in the generalization of eq. (20b):

\[ \frac{\delta}{\delta \Phi_i} D^J_{jk} = \frac{\delta^2 \Gamma[\Phi]}{\delta \Phi_j \delta \Phi_k}^{-1} = - \left( \frac{\delta^2 \Gamma[\Phi]}{\delta \Phi_i \delta \Phi_m} \right)^{-1} \left( \frac{\delta^2 \Gamma[\Phi]}{\delta \Phi_m \delta \Phi_k} \right) = D^J_{jm} \Gamma^J_{mn} D^J_{nk}. \quad (29) \]

It means that the derivative of a propagator consists of a sum of three-point functions to which two propagators are attached. Graphically this is depicted in fig. 2, where the superfield is represented by double lines. There also the following differentiation rules are shown:

\[ \frac{\delta}{\delta \Phi}, \Gamma^J_{j_1 \ldots j_n} = - \frac{\delta W[J]}{\delta \Phi_i} \frac{\delta^2 \Gamma[\Phi]}{\delta \Phi_j \delta \Phi_k}, \quad (30) \]

\[ \frac{\delta}{\delta \Phi} \Gamma^J_{j_1 \ldots j_n} = - \frac{\delta W[J]}{\delta \Phi_i} \frac{\delta^2 \Gamma[\Phi]}{\delta \Phi_j \delta \Phi_k}. \quad (31) \]

The starting equation for the DSEs of 1PI functions reads

\[ - \frac{\delta S}{\delta \phi_i} \big|_{\phi_i=\Phi_i+D^J_{j} \delta / \delta \phi_j} + \frac{\delta \Gamma[\Phi]}{\delta \Phi_i} = 0. \quad (32) \]

It is helpful to have this equation worked out for generic fields, that means we just use the superfield for the derivatives. Doing so we take into account all possibilities and can use this as starting point for individual cases. This equation is depicted in fig. 3. The ensuing two-point DSE is shown in fig. 4.

The differentiation rules in fig. 2 can directly be used in the derivation of DSEs once one has computed the generating equation (32). Details on the formalism can be found in [3, 6]. Here I only would like to demonstrate with a further example the usefulness of the graphical approach. It will also demonstrate another feature of DSEs: If there is more than one field, there are several versions of the same equation.

C. Another example: The ghost-gluon vertex of Landau gauge Yang-Mills theory

A theory investigated intensively with DSEs is Landau gauge Yang-Mills theory. The action can be written as

\[ S[A, \bar{c}, c] = \frac{1}{2} S_{ij} A_i A_j + S_{ij} \bar{c}_i c_j + \frac{1}{2} S_{ijkl} A_i A_j A_k - \frac{1}{4} S_{ijklm} A_i A_j A_k A_l - S_{ij,kl} A_i A_j \bar{c}_k c_l. \quad (33) \]

The indices comprise color and Lorenz indices and the space dependence. The explicit form is not given, as it is not required for the demonstration purposes here. In the following gluons are denoted by wiggly and ghosts by dashed lines. Superfields again are represented by double lines.
\[
\frac{\delta}{\delta \phi_1} \times \frac{\delta}{\delta \phi_i} = \frac{\delta}{\delta \phi_1} \frac{\delta}{\delta \phi_i}
\]

FIG. 2: Diagrammatic rules for differentiating an external field, a propagator or a vertex. The circle with the cross denotes an external field, small blobs denote dressed propagators, and big blobs dressed 1PI vertices. The double line represents the super-field \( \Phi \).

\[
\frac{\delta \Gamma}{\delta \phi_i} = \begin{array}{c}
-1/2 \times \times \\
-1/2 \times \\
-1/3! \\
-1/2 \times \\
-1/2 \times \\
-1/3!
\end{array}
\]

FIG. 3: The DSE for 1PI functions. Crosses in circles denote external fields. All internal propagators are dressed and the big blob denotes a dressed 1PI vertex function.

Let us derive the DSEs of the ghost-gluon vertex of Landau gauge QCD. There are two distinct versions, because the form of the equations depends on the first derivative: If we differentiate with respect to a (anti-)ghost field, we get another equation as when we differentiate with respect to a gluon field. The reason is simply that the first differentiation determines which bare vertices can appear. In the former version there are only bare ghost-gluon vertices and in the latter all bare vertices appear.

Let us start with the simpler version, i.e., the first derivative is done with respect to a ghost field. Replacing the fields as given in eq. (32) and differentiating with respect to an anti-ghost yields

The easiest way to work this out is to use fig. 4 and take the external field as ghosts fields. The symmetry factor 1/2 in the third diagram vanished because there were two possibilities to plug the bare ghost-gluon vertex into the diagram. In the present case the general diagrammatic rules from fig. 2 specialize to

\[
\begin{array}{c}
-1 \\
-1/2 \times \\
-1/2 \\
-1/3! \\
-1/2 \\
-1/2 \\
-1/2
\end{array}
\]

FIG. 4: The DSE for a generic two-point function, sources not set to zero yet.
Performing the remaining derivative with respect to a gluon leads to

\[
\frac{\delta}{\delta A_i} \langle \mathcal{M}\rangle = \frac{1}{2} \frac{\delta}{\delta A_i} \langle \mathcal{M}\rangle + \frac{1}{2} \frac{\delta}{\delta A_i} \langle \mathcal{M}\rangle + \frac{1}{3!} \frac{\delta}{\delta A_i} \langle \mathcal{M}\rangle.
\]

The chosen convention for the definition of vertices induced a change of sign. When we set the external sources to zero, the mixed propagators become irreducible gluon and ghost propagators. The pure super-field propagator in the second and third terms on the right-hand side yields a sum of different terms, when decomposed, but ghost number conservation allows here only vertices with the same number of ghost and anti-ghost legs. Therefore, for each diagram only one propagator can be realized. The final result is then the ghost-gluon vertex DSE:

\[
\frac{\delta}{\delta A_i} \langle \mathcal{M}\rangle = \frac{1}{2} \frac{\delta}{\delta A_i} \langle \mathcal{M}\rangle + \frac{1}{2} \frac{\delta}{\delta A_i} \langle \mathcal{M}\rangle + \frac{1}{3!} \frac{\delta}{\delta A_i} \langle \mathcal{M}\rangle.
\]

The second version of the ghost-gluon DSE is obtained if we start differentiating with respect to the gluon field. In this case the super-field is important as will become evident below. For brevity we skip diagrams that do not contribute to the ghost-gluon vertex (the tadpole and all incompatible tree graphs as well as the graph with the bare four-gluon vertex connected to an external field). The factor in front of the loop containing ghost-fields is changed from 1/2 to 1, because there are two possibilities to insert the bare ghost-gluon vertex as we have to consider the direction of fermion lines explicitly. The diagrams left in fig. 4 are

\[
\frac{\delta}{\delta \bar{A}_i} \langle \mathcal{M}\rangle = \frac{1}{2} \frac{\delta}{\delta \bar{A}_i} \langle \mathcal{M}\rangle + \frac{1}{2} \frac{\delta}{\delta \bar{A}_i} \langle \mathcal{M}\rangle + \frac{1}{3!} \frac{\delta}{\delta \bar{A}_i} \langle \mathcal{M}\rangle.
\]

Differentiation with respect to the anti-ghost field yields
Again propagators partly involving the super-field are determined by the second field and pure super-field propagators by the symmetries of the vertices. Finally the second version of the ghost-gluon vertex DSE is obtained by setting the external sources to zero:

Here one can see the importance of the super propagator, because otherwise we would have missed contributions like the first diagram in the second line.