Asymptotically Safe Cosmology

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Well-established standard model of cosmology
Asymptotic safety
Modifications by running couplings
Fixed point regime
Standard cosmology

- Hubble parameter $H$
- Ideal fluids with energy density $\rho_i$ and equation of state $p_i = \omega_i \rho_i$
- Scalar field $\phi$, $\rho_\phi = \frac{\dot{\phi}^2}{2} + V$, $p_\phi = \frac{\dot{\phi}^2}{2} - V$

\[ \begin{align*}
\dot{H} &= -\frac{\kappa^2}{2} \left( \sum_i (1 + \omega_i) \rho_i + \dot{\phi}^2 \right) \\
\dot{\rho}_i &= -3(1 + \omega_i) H \rho_i \\
\ddot{\phi} &= -3H \dot{\phi} - \frac{dV}{d\phi} \\
H^2 &= \frac{\kappa^2}{3} \left( \sum_i \rho_i + \frac{\dot{\phi}^2}{2} + V \right) \Rightarrow 1 = \sum_i \Omega_i + x^2 + y^2 = \sum_i \Omega_i + \Omega_\phi
\end{align*} \]
Dimensionless variables

\[ \Omega_i = \frac{\kappa^2 \rho_i}{3H^2}; \quad x = \frac{\kappa \dot{\phi}}{\sqrt{6}H}; \quad y = \frac{\kappa \sqrt{V}}{\sqrt{3}H}; \quad z = \frac{V'}{\kappa V}; \quad \eta = \frac{V''}{\kappa^2 V}; \quad N = -\ln a \]

\[
\frac{dx}{dN} = 3x(1 - x^2) + \sqrt{\frac{3}{2}} y^2 z - \frac{3}{2} x \sum_i (1 + \omega_i) \Omega_i \\
\frac{dy}{dN} = -xy \left(3x + \sqrt{\frac{3}{2}} z\right) - \frac{3}{2} y \sum_i (1 + \omega_i) \Omega_i \\
\frac{dz}{dN} = -\sqrt{6} x(\eta - z^2), \\
\frac{d\Omega_i}{dN} = -3\Omega_i \left(2x^2 + \sum_j \gamma_j \Omega_j - \gamma_i\right)
\]
Flow diagram for exponential potential and $\omega = 0, z=1$
## Cosmological fixed points

<table>
<thead>
<tr>
<th>Case</th>
<th>( x )</th>
<th>( y )</th>
<th>( z )</th>
<th>( \Omega \gamma )</th>
<th>existence</th>
<th>type</th>
</tr>
</thead>
<tbody>
<tr>
<td>(a)</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>all ( \gamma )</td>
<td>potential</td>
</tr>
<tr>
<td>(b)</td>
<td>( \pm 1 )</td>
<td>0</td>
<td>( z_* )</td>
<td>0</td>
<td>all ( \gamma ) and ( z_* )</td>
<td>kinetic</td>
</tr>
<tr>
<td>(c)</td>
<td>( -\frac{z_*}{\sqrt{6}} )</td>
<td>\ inability to render properly</td>
<td>( z_* )</td>
<td>0</td>
<td>( z_*^2 \leq 6 )</td>
<td>mixed</td>
</tr>
<tr>
<td>(d)</td>
<td>( -\sqrt{\frac{3\gamma}{2}} z_* )</td>
<td>\ inability to render properly</td>
<td>( z_* )</td>
<td>( 1 - \frac{3\gamma}{z_*^2} )</td>
<td>( z_*^2 \geq 3\gamma )</td>
<td>scaling</td>
</tr>
<tr>
<td>(e)</td>
<td>0</td>
<td>0</td>
<td>( - )</td>
<td>1</td>
<td>( \gamma \neq 0 )</td>
<td>fluid</td>
</tr>
</tbody>
</table>

\( z_* \) is a solution to the equation \( \eta(z) = z^2 \), \( \gamma = 1 + \omega \)
Fixed point behaviour

- Various fixed points where
  - either the fluid component dominates,
  - or the scalar field dominates (either only the kinetic term or the potential or a mixture of both)
  - or scaling solution (neither fluid nor scalar field dominate)

- At a fixed point the equation of state parameter for the scalar field becomes constant

\[
1 + \omega_\phi = 1 + \frac{p_\phi}{\rho_\phi} = \frac{2x^2}{x^2 + y^2}
\]

\[
\frac{d \ln H}{dN} = 3x^2 + \frac{3}{2}(1 + \omega)\Omega \Rightarrow < 1 \text{ for inflation}
\]

- Attractivity properties of the fixed point can lead naturally to accelerated expansion
RG modified cosmology

\[ G \rightarrow G_k, \ \Lambda \rightarrow \Lambda_k, \ \mathcal{V} \rightarrow \mathcal{V}_k \]

RG scale dependence is given in terms of

\[ \eta_{\text{RG}} = \frac{\partial \ln G_k}{\partial \ln k}, \ \nu_{\text{RG}} = \frac{\partial \ln V_k}{\partial \ln k}, \ \sigma_{\text{RG}} = \frac{\partial \ln V'_k}{d \ln k} \]

At renormalization group fixed point

\[ \eta_{\text{RG}} = -2; \ \nu_{\text{RG}} = 4; \ \sigma_{\text{RG}} = 3 \]

Cosmological constant included in the potential
RG modified cosmology

- RG effects become important at early times
- Assume $k = k(t) = k(N)$
- In many papers: $k \propto H$

\[
\frac{dx}{dN} = 3x(1 - x^2) + \sqrt{\frac{3}{2}} y^2 z - \frac{3}{2} x \sum \gamma_i \Omega_i + \frac{1}{2} x \eta_{\text{RG}} \frac{d \ln k}{dN}
\]

\[
\frac{dy}{dN} = -\sqrt{\frac{3}{2}} xyz - 3x^2 y - \frac{3}{2} y \sum \gamma_i \Omega_i + \frac{1}{2} y (\eta_{\text{RG}} + \nu_{\text{RG}}) \frac{d \ln k}{dN}
\]

\[
\frac{dz}{dN} = -\sqrt{6} x (\eta - z^2) + z \left( -\frac{1}{2} \eta_{\text{RG}} - \nu_{\text{RG}} + \sigma_{\text{RG}} \right) \frac{d \ln k}{dN}
\]

\[
\frac{d\Omega_i}{dN} = -3 \Omega_i \left( 2x^2 + \sum_{j} \gamma_j \Omega_j - \gamma_i \right) + \eta_{\text{RG}} \Omega_i \frac{d \ln k}{dN}
\]
Bianchi identity

\[ \nabla^\mu G_{\mu\nu} = \nabla^\mu (8\pi G_k T_{k\mu\nu}) = 0 \]

From the Friedmann constraint follow

\[ \frac{d\ln k}{dt} \frac{\partial (G_k \rho_k)}{\partial \ln k} = 0 \Rightarrow \frac{\partial H}{\partial \ln k} = 0 \]

\[ \eta_{RG}(k) + y^2 \nu_{RG}(k, z) = 0 \]
**RG scale and time dependence**

\[
\frac{d \ln k}{dN} = \frac{1}{\alpha_{\text{RG}}} \left[ \frac{\sigma_{\text{RG}}}{\nu_{\text{RG}}} \sqrt{\frac{3}{2}} x z + 3 x^2 + \frac{3}{2} \sum_i \gamma_i \Omega_i \right]
\]

\[
\frac{d \ln H}{dN} = \alpha_{\text{RG}} \frac{d \ln k}{dN} - \frac{\sigma_{\text{RG}}}{\nu_{\text{RG}}} \sqrt{\frac{3}{2}} x z
\]

\[
\alpha_{\text{RG}} = \frac{1}{2} \left[ \eta_{\text{RG}} + \nu_{\text{RG}} - \frac{\partial}{\partial \ln k} \ln \left( -\frac{\eta_{\text{RG}}}{\nu_{\text{RG}}} \right) \right]
\]

- At RG fixed point: \( \alpha_{\text{RG}} = 1 \)
- \( k \propto H \) for \( \alpha_{\text{RG}} = 1 \) with either \( \sigma_{\text{RG}} = 0, 1/\nu_{\text{RG}} = 0, x = 0 \) or \( z = 0 \)
- In general, \( k \propto H \) does not hold because of the nontrivial \( k \)-dependence
Types of fixed points

- In $k$ may or may not be forced to be constant with scale factor

Simultaneous fixed points

$$\frac{d \ln k}{dN} = \text{const.} \neq 0$$

- $k$ keeps evolving with $N$
- Fixed points are only reached together with RG fixed points because of implicit and explicit scale dependence

Freeze-in fixed points

$$\frac{d \ln k}{dN} = 0$$

- Evolution of $\ln k$ with $N$ comes to a halt at some freeze-in scale $k_{fi}$
- $\frac{d \ln H}{d \ln N}$ can still be non-zero
- RG couplings can take generic values different from fixed point values
### Renormalization group modified fixed points

<table>
<thead>
<tr>
<th>Case</th>
<th>$x$</th>
<th>$y$</th>
<th>$z$</th>
<th>$\frac{d \ln k}{d N}$</th>
<th>existence</th>
</tr>
</thead>
<tbody>
<tr>
<td>(a)</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>-</td>
<td>$- \frac{\eta_{RG}}{\nu_{RG}} = 1$</td>
</tr>
<tr>
<td>(b)</td>
<td>±1</td>
<td>0</td>
<td>$z_*$</td>
<td>$\frac{6}{\nu_{RG}} \left( 1 \pm \frac{1}{2} \left( \frac{\sigma_{RG}}{\nu_{RG}} \right) \sqrt{6}z_* \right)$</td>
<td>$\eta_{RG} = 0$</td>
</tr>
<tr>
<td>(c1)</td>
<td>$\pm \sqrt{1 + \frac{\eta_{RG}}{\nu_{RG}}}$</td>
<td>$\sqrt{-\frac{\eta_{RG}}{\nu_{RG}}}$</td>
<td>0</td>
<td>$\frac{6}{\nu_{RG}}$</td>
<td>$\eta(0) = 0$</td>
</tr>
<tr>
<td>(c2)</td>
<td>$\pm \sqrt{1 + \frac{\eta_{RG}}{\nu_{RG}}}$</td>
<td>$\sqrt{-\frac{\eta_{RG}}{\nu_{RG}}}$</td>
<td>$z_*$</td>
<td>$\frac{6}{\nu_{RG}} \left( 1 + \frac{z_<em>}{\sqrt{6}x_</em>} \right)$</td>
<td>$\sigma_{RG} = \nu_{RG}$</td>
</tr>
<tr>
<td>(d1)</td>
<td>0</td>
<td>$\sqrt{-\frac{\eta_{RG}}{\nu_{RG}}}$</td>
<td>0</td>
<td>$\frac{3\gamma}{\nu_{RG}}$</td>
<td>$\gamma \neq 0$</td>
</tr>
<tr>
<td>(d2)</td>
<td>$\pm \sqrt{-\frac{\eta_{RG}}{\nu_{RG}} \frac{\gamma}{2 - \gamma}}$</td>
<td>$\sqrt{-\frac{\eta_{RG}}{\nu_{RG}}}$</td>
<td>$-\sqrt{\frac{3\gamma}{2x_*}}$</td>
<td>0</td>
<td>$\sigma_{RG} = \nu_{RG}$</td>
</tr>
</tbody>
</table>

Cases (a), (b), (c): $\Omega_\gamma = 0$, (d1) $\Omega_\gamma = 1 - y_*^2$, (d2) $\Omega_\gamma = 1 - 2y_*^2(2 - \gamma)^{-1}$

In case (b), $z_*$ is a solution to $-\eta(z) + z^2 \left( \frac{\sigma_{RG}}{\nu_{RG}} \right)^2 \mp \sqrt{6} \left( \frac{\sigma_{RG}}{\nu_{RG}} \right) z = 0$,
in case (c2) to $-\eta(z) + z^2 (1 + R) + \sqrt{6}x_* R z = 0$ where $R = y_*^2/(2x_*^2)$
### Case (a) (potential): inflation, $\eta_{RG} = -\nu_{RG}$

- **Simultaneous fixed point:**
  - $\eta_{RG} = -2$, from e.g. $V(\phi) = \lambda_2 \phi^2 \Rightarrow z = 2/(\kappa \phi) = 0 \Rightarrow \phi \to \infty$
  - $\nu_{RG} = \sigma_{RG} = \beta_2/\lambda_2$, where $\beta_2$ is the $\beta$-function for $\lambda_2$
  - $\nu_{RG} = 2$ requires $\lambda_2 \to k^2 \tilde{\lambda}_2$ and $\tilde{\lambda}_2 = \text{const.} \neq 0$

- **Freeze in:**
  - Fixed point values entail $d \ln H/dN = 0$, $\alpha_{RG} \neq 0$
  - $\frac{\partial}{\partial \ln k} \ln(-\eta_{RG}/\nu_{RG})$ cannot vanish at freeze-in, $\nu_{RG}$ and $\eta_{RG}$ cannot simultaneously achieve fixed points

### Case (b) (kinetic): $\eta_{RG} \to 0$ and $\nu_{RG} \neq 0$

- **Essentially classical near Gaussian fixed point of Einstein gravity**

- **Simultaneous fixed point:**
  - $V = V_0 \exp(\lambda \kappa \phi)$ with $\lambda = \mp \sqrt{6}/((\sigma_{RG}/\nu_{RG} + 1)$ or
  - $V = V_0 (1 - \exp \lambda Q \kappa \phi)^{-1/Q}$ where $Q = (\sigma_{RG}/\nu_{RG})^2 - 1$.

- **Freeze-in:** general potential and $z = \mp (2\nu_{RG})/(\sqrt{6}\sigma_{RG})$
Mixed solution fixed points

Case (c1) (mixed) simultaneous:
- E.g. with monomial potential $\lambda n\phi^n$ and $\phi \to \infty$
- For Gaussian matter fixed point with $V \to \lambda_0$: equipartition between kinetic and potential energy: $x^2 = y^2 = \frac{1}{2}$

Case (c2) mixed:
- Simultaneous:
  - $V = V_0 (1 - \exp \lambda Q \kappa \phi)^{-1/Q}$ with $\lambda = \sqrt{6}x_*$ and $Q = y_*/(2x_*)$
- Freeze in:
  - $z_* = -\sqrt{6}x_*$, $\eta = z^2$, $\nu_{RG} = \sigma_{RG}$
  - $V = V_0 \exp(-\sqrt{6}x_* \kappa \phi)$
  - Inflation if $\frac{\eta_{RG}}{\nu_{RG}} < -\frac{2}{3}$
Scaling solution fixed points

Case (d1) (scaling): simultaneous with $\gamma \neq 0$

- $\eta_{RG} = -2$, $\nu_{RG} = 4$, $\alpha_{RG} = 1$ and only cosmological constant:
  - $x = 0$ and $y^2 = 1/2$, $H \propto k$ at the fixed point

Case (d2) (scaling): freeze in, $\eta(z) = z^2$, $V = V_0 \exp(\lambda \kappa \phi)$

- $\lambda = -\sqrt{\frac{3\gamma(2-\gamma)}{2}} \left( -\frac{\nu_{RG}}{\eta_{RG}} \right)$, inflation for $\gamma < 2/3$
Summary and Outlook

- RG effects transmitted via quantities $\eta_{RG}$, $\nu_{RG}$, $\sigma_{RG}$
- Consistency equation to preserve the Bianchi identity
- Modified fixed points of cosmological evolution
- Universe may accelerate at the fixed points

- How does the universe evolve toward or away from the fixed points
- Constrain transition scale by observations