Renormalization Group Study of the Chiral Phase Transition

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The aim:
full quantum action $\Gamma$

Starting point: classical, bare action $S$ at an ultraviolet scale $\Lambda$

*Step by step* integration of fluctuations in momentum shells

Scale dependent effective action $\Gamma_k$

Flow equation: the change of the average effective action with the scale $k$
Wetterich equation

- Here: one component scalar field $\varphi(x)$
- Introduce scale dependence with a regulator term in the generating functional

$$Z_k[J] = e^{W_k[J]} = \int D\varphi \, e^{-S[\varphi] - \Delta S_k[\varphi]} + \int J\varphi$$

- Regulator term is quadratic in fields

$$\Delta S_k[\varphi] = \frac{1}{2} \int \frac{d^Dp}{(2\pi)^D} \varphi(-p)R_k(p)\varphi(p)$$

- $R_k(p)$ satisfies certain criteria (optimized regulator!)

[D. F. Litim, J. M. Pawlowski (2002)]
Wetterich equation

- Effective average action via modified Legendre transform

\[
\Gamma_k[\phi] = \sup_J \left( \int J\phi - W_k[J] \right) - \Delta S_k[\phi]
\]

(3)

- Wetterich equation is

\[
\partial_t \Gamma_k = \frac{1}{2} \text{Tr} \left\{ \partial_t R_k \left[ \Gamma_k^{(2)}[\phi] + R_k \right]^{-1} \right\}
\]

(4)

- IR regularization: \( \lim_{p^2/k^2 \to 0} R_k(p) > 0 \)

- UV regularization: \( \partial_t R_k(p) \) is peaked around \( p \approx k \)

[H. Gies (2006)]
Proper-time (PT) flow equation

- One-loop expression for effective action

\[ \Gamma^{1-\text{loop}}[\phi] = S[\phi] + \frac{1}{2} \text{Tr} \ln S^{(2)}[\phi] \]  (5)

- Schwinger proper-time representation of the logarithm and regulator function \( f(\Lambda, s) \)

\[ \Gamma^{1-\text{loop}}[\phi] = S[\phi] - \frac{1}{2} \int \frac{ds}{s} f(\Lambda, s) \text{Tr} \exp \left( -sS^{(2)} \right) \]  (6)

- Scale dependence: \( f(\Lambda, s) \rightarrow f_k(\Lambda, s) = f(s\Lambda^2) + f(sk^2) \)

- Last step: RG improvement

\[ \partial_t \Gamma_k = -\frac{1}{2} \int_0^\infty \frac{ds}{s} \left( \partial_t f(sk^2) \right) \exp \left( -s\Gamma_k^{(2)} \right) \]  (7)
PT flow as a approximation to Wetterich flow

- Class of optimized regulator functions

\[
\partial_t f^{(m)}(sk^2) = -\frac{2}{\Gamma(m)}(sk^2)^m e^{-sk^2}
\]  

(8)

- Flow simplifies to

\[
\partial_t \Gamma_k = \text{Tr} \left[ (\Gamma_k^{(2)} + k^2)^{-1} k^2 \right]^m
\]  

(9)

- Generalized proper time flows in the background formalism

\[
\partial_t \Gamma_k^{\text{Wetterich}} = \partial_t \Gamma_k^{\text{sPTRG}} + \partial_t \Gamma_k^{(2)}
\]  

(10)

- Term \(\partial_t \Gamma_k^{(2)}\) vanishes at the criticality

- In the leading order of derivative expansion it is possible to map Wetterich equation regulator \(\rightarrow\) PT flow regulator

- Mapping is injection: \(m \leq d/2 + 1\) (d is the number of dimensions)
Scales of QCD

- Above 2 GeV perturbative QCD
- At lower momentum scales: bound states, quark condensate, confinement...
- Possible hierarchy of the scales:
  - Compositeness scale
    \( k_\phi \approx 1 \text{ GeV} \) - mesonic bound states are formed
  - Chiral symmetry breaking scale
    \( k_\chi = 500 \text{ MeV} \)
  - Baryon formation

![Diagram showing scales of QCD](image-url)
Motivation for Quark-Meson Model

- Start the RG evolution at $\Lambda = k_\phi$
- Initial effective action: two flavor quark-meson model

\[
\Gamma = \int d^4x \left\{ \frac{1}{2} (\partial_\mu \phi)^2 + U(\phi) + \bar{q} [\gamma \partial + g (\sigma + i \vec{\pi} \tau_5) ] q \right\}
\]  

(11)
- Short-hand notation $\phi = (\sigma, \vec{\pi})$
- $U(\phi)$ meson potential - Mexican hat potential
- Physical picture behind:
  - integrate out all gluons
  - non-local effective 4 and higher quark vertices are replaced with mesons
- Truncation: only scalar and pseudoscalar channel
- Most important for thermodynamics
- Baryons are omitted: good approximation for small quark chemical potentials
Flow for Quark-Meson Model

- Proper-time flow equation with optimized regulator function $f_k^{5/2}(sk^2)$
- Lowest order of derivative expansion

$$\Gamma[\phi] = \int d^4x \left\{ U(\phi) + \frac{1}{2} Z (\partial_\mu \phi)^2 + \ldots \right\}$$

(12)

with $Z = 1$
- Projection on constant fields $\phi(x) = (\sigma, \vec{0})$
- Yukawa $g$ coupling is not running
- Temperature - imaginary time formalism
- And finite quark chemical potential

$$\Gamma = \int_0^\beta d\tau \int d^3x \left\{ \frac{1}{2} (\partial_\mu \phi)^2 + U(\phi) + \bar{q} \left[ \gamma \partial + g(\sigma + i\vec{\pi}\vec{\tau}\gamma_5) + \gamma_0 \mu \right] q \right\}$$

(13)

- Isospin symmetry is assumed
- Flow equation for the grand potential $U_k$
Interpretation

- Flow equation:

\[
\partial_t U_k(T, \mu) = \frac{k^5}{12\pi^2} \left( \frac{1}{E_\sigma} \coth \frac{E_\sigma}{2T} + \frac{3}{E_\pi} \coth \frac{E_\pi}{2T} - \frac{2N_c N_f}{E_q} \tanh \frac{E_q - \mu}{2T} \right.
\]

\[
\left. - \frac{2N_c N_f}{E_q} \tanh \frac{E_q + \mu}{2T} \right) \tag{14}
\]

- quark/antiquark \( E_q = \sqrt{k^2 + g^2 \phi^2} \)

- pion \( E_\pi = \sqrt{k^2 + 2U'_k} \)

- \( \sigma \)-meson \( E_\sigma = \sqrt{k^2 + 2U'_k + 4\phi^2 U''_k} \)

- Additive contribution of relevant degrees of freedom to the equation

- \( T = 0 \): Silver Blaze property

\[
\partial_t U_k = \frac{k^5}{12\pi^2} \left( \frac{1}{E_\sigma} + \frac{3}{E_\pi} - \frac{4N_c N_f}{E_q} \Theta(E_q - \mu) \right) \tag{15}
\]

- Opposite roles of quarks and meson in RG evolution
Comparison with Mean-Field (MF) Approximation

- **MF approximation:**
  - neglecting meson fluctuations
  - replacing meson fields replaced with their expectational values \((\sigma_0, \vec{0})\)

- **Grand potential for the MF**

\[
U = U_B(\sigma_0, \vec{0}) + \nu T \int \frac{d^3 p}{(2\pi)^3} \left\{ -\beta E_q + \ln[1 - n_q(T, \mu)] + \ln[1 - n_{\bar{q}}(T, \mu)] \right\}, \quad (16)
\]

where

\[
n_q(T, \mu) = \frac{1}{1 + \exp[\beta(E_q - \mu)]}, \quad n_{\bar{q}}(T, \mu) = n_q(T, -\mu) \quad (17)
\]

- Cross out the meson potential, find scale derivative

- **Flow in the MF approximation**

\[
\partial_t U_k(T, \mu) = -\frac{k^5}{12\pi^2} \frac{2N_c N_f}{E_q} \left( \tanh \frac{E_q - \mu}{2T} + \tanh \frac{E_q + \mu}{2T} \right) \quad (18)
\]

- Flow for QM model enables studying the influence of the meson fluctuations
Taylor Expansion of the Effective Potential

- Start at UV scale: Mexican hat potential
- integrating out fluctuations $\rightarrow$ generating new vertices
- couplings are scale dependent coefficients in expansion of $U_k$
- notation: $\phi^2 = \rho$

**Symmetric phase**

\[ U_k = \sum_{n=0}^{N} \frac{a_n}{n!} \rho^n \]  

\[ \frac{\partial U(0)}{\partial \rho} < 0 \rightarrow \text{non-trivial minimum } \rho_0 \]

\[ \partial_t U_k \rightarrow N \text{ beta functions for } \{a_0, \ldots, a_N\} \]

- Small number $N$ of equations
- $U_k$ known only at its minimum

**Broken symmetry phase**

\[ U_k = b_0 + \sum_{n=2}^{N} \frac{b_n}{n!} (\rho - \rho_0)^n \]  

\[ \partial_t U_k \rightarrow \text{beta functions for } \{b_0, b_2, \ldots, b_N\} \]

- additional $\partial_t \rho_0$
Initial ultraviolet parameters

- Initial effective potential in vacuum for UV scale $k_\Lambda$
  \[
  U_\Lambda(\phi) = \frac{\lambda}{4}(\phi^2 - v^2)^2 - c\sigma 
  \]  
  (21)
- Initial parameters give $f_\pi = \sigma_0 = 93 \text{ MeV}$ for $k = 0$ and chiral symmetry breaking scale in vacuum
- Yukawa coupling: $m_q = g\sigma_0$
- Initial UV parameters fixed for finite $T$ and $\mu$
- Temperature range is limited $T/\Lambda < 0.1$
Influence of the Truncation Order

- Results for $\mu = 0$
- Truncation orders $N = 3, 4, 5$
- Convergence for $N = 5$
- Note: $k_\chi$ independent of the truncation order
- Second order phase transition at $T_c = 132.0$ MeV with $N = 5$
Mass Sensibility for Taylor Expanded Potential

- Chiral limit: $T_c = 132.0$ MeV
- Pions - Goldstone bosons
- Explicit $\chi$SB: a crossover
- $T > 200$ MeV: pions and $\sigma$-meson acquire the same mass
- Meson masses rise linearly with $T > 200$ MeV - lowest quark Matsubara mode
- Influence of fixed initial potential parameters
Effective Potential on a Grid

- Problem: \( \partial_t U(\sigma_i) = F(U'(\sigma_i), U''(\sigma_i)) \)
- Also, \( \partial_t U'(\sigma_i) = G(U''(\sigma_i), U^{(3)}(\sigma_i)) \) and so on ...
- Find derivatives \( U''(\sigma_i) \) and \( U^{(3)}(\sigma_i) \) independently of the flow

- 1D grid with \( M \) \( \sigma \)-field points
- \( M > N \) flow equations
- Global minimum of \( U_k \)

Taylor expansion of \( U(\sigma_i) \) at each grid point to the third order
- Matching expansions at half distance between neighboring grid points
RG Evolution on Grid

\[ (T, \mu) = (100 \text{ MeV}, 0) \]
\[ k = 950 \text{ MeV} \]
\((T, \mu) = (100 \text{ MeV}, 0)\)
\(k = 850 \text{ MeV}\)
\( \sigma \) [MeV]

\( U \) [MeV^4]

\((T, \mu) = (100 \text{ MeV}, 0)\)
\( k = 750 \text{ MeV} \)
\( (T, \mu) = (100 \text{ MeV}, 0) \)

\( k = 650 \text{ MeV} \)
$(T, \mu) = (100 \text{ MeV}, 0)$

$-k = 550 \text{ MeV}$

$U \ [\text{MeV}^4]$ vs $\sigma \ [\text{MeV}]$
Spontaneous symmetry breaking
Minimum of $U_k$ is fixed

Inner part of $U_k$ flattens
Minimum of $U_k$ is fixed

Inner part of $U_k$ flattens
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Minimum of $U_k$ is fixed

Inner part of $U_k$ flattens
Minimum of $U_k$ is fixed

Inner part of $U_k$ flattens
Chiral limit: \( T_c = 133.5 \text{ MeV} \)

Explicit \( \chi \text{SB: crossover} \)

Comparison with the Taylor expanded potential ...
Comparison

- **Chiral limit:**

- **Explicit χSB**
Summary

- Quark-meson model as a good approximation to QCD
- Flow equation
- Comparison of two solution strategies
- To do: map phase diagram for $\mu > 0$