A phase space approach to closed orbits in central potentials

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Motivated by attempts to understand celestial mechanics, studying the motion of particles in conservative central force fields is one of the oldest subjects counted among physics. Astronomical observations of planetary motion reveal practically closed orbits, i.e. planets retracing their own paths. A question naturally arising is, whether this is a general feature of central force fields or a special additional symmetry for certain force laws. Analytic solutions of the differential equation for the orbit only exist for a handful of force laws. However, analyzing central force motion by means of classical mechanics allows to draw many conclusions about the nature of orbits without explicitly solving the equations of motion. Requiring conservation of energy and angular momentum imposes some elementary symmetry properties on the orbit, making it possible to find a mathematical condition for closedness of orbits. Translating this condition into a condition for the force law, a theorem first proved in 1873 by J. Bertrand can be derived. Bertrand’s theorem states, that there are only two force laws resulting in closed orbits for all bound particles: the inverse-square law and Hooke’s law. A particularly descriptive treatment of this theorem and orbits in central force fields in general is gained introducing a phase space approach.

According to the well-known solution of the Kepler problem assuming Newton’s law of universal gravitation, planetary orbits are ellipses and hence closed curves. The question, if this is the only force law imposing closed orbits is answered by Bertrand’s theorem. Closed orbits are possible for all attractive central potentials, but there are only two potentials resulting in closed orbits for all bound states: the harmonic oscillator potential and potentials varying as $1/r$ such as the Newtonian or the Coulomb potential. An orbit is said to be closed, if it retraces itself after a finite number of revolutions around the center of force. Among all imaginable curves satisfying this condition, only a particular class can be produced by central fields, due to the laws of classical mechanics. Owing to these limitations on possible orbits, the before stated question becomes solvable. Starting from essentially identical differential equations for the orbit, the various proofs of the theorem mentioned above have in common the main steps of argumentation following physical considerations while partly differ widely in the mathematical implementation. The original demonstration by French mathematician Joseph Bertrand of 1873 [1] (also available in a translated version [2]) is an interesting mathematical proof, as such, however not putting a lot of emphasis on the physical meaning of quantities introduced or assumptions made. Another demonstration presented in [3] is, like many others, completely based on treating closed orbits as perturbed circular orbits. Therefore, the condition for potentials found in this way is only necessary but not sufficient. Explicitly checking the remaining two possible types of potential for closedness of orbits leads to the same statement, yet. Another proof more accessible to readers less familiar with perturbation theory is presented in [4] using the ideas of [5], following an especially descriptive phase space approach. This article aims at sketching this alternative proof of Bertrand’s theorem focusing on the basic assumptions made and the phase space depiction of orbits in central fields. After introducing a phase space portrait representing orbits and interpreting its elementary features, a closedness-condition is formulated in terms of an area in phase space and evaluated in a case analysis revealing the theorem. Intermediately, emphasis is put on two key points common to all proofs known to the author. Putting it in a more general context, Bertrand’s theorem can be looked on as solving an inverse problem. The conventional central force problem is concerned with finding orbits resulting from a certain potential. As is shown here, it is also possible to fix potentials by the orbits they produce. For example, knowing from observation the shape of orbits in gravitational fields directly yields the functional form of the law of gravitation.

In order to mathematically treat the motion of a particle in a central potential $V(r)$, it is convenient to make use of conservation laws, namely conservation of energy and angular momentum. Taking advantage of these quantities conserved in central force fields, a particularly useful differential equation for the orbit in polar coordinates $r$ and $\theta$ in terms of $u = 1/r$ can be obtained [4]:

$$\frac{1}{2} \left( \frac{du}{d\theta} \right)^2 + \frac{1}{2} u^2 + \alpha V \left( \frac{1}{u} \right) = E' \quad (1)$$

where $\alpha = m/\ell^2$ and $E' = \alpha E$ ($m$ being the mass, $E$ the energy and $\ell$ the magnitude of the angular momentum of the particle). Interpreting (1) as the equation of a curve in the $u$-$\theta'$ plane, a phase space with coordinates $u$ and $u' := \frac{du}{d\theta}$ can be introduced. Thus, every orbit $r(\theta)$

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1 Here the terms central potential and central force law are partly used synonymous. The relation between those two physical quantities is given by $F(r) = \frac{d}{dr}V(r)$, where $F$ is the magnitude of the force and $V$ the potential.

2 At least at one point all proofs known to the author involve slightly perturbed circular orbits.
FIG. 1: This is a schematic depiction of the phase space portrait of a bound orbit defined by equation (1). It is symmetric about the $u$-axis, crossing it twice corresponding to motion between a maximum and a minimum value of the distance from center. The area enclosed between the $u$-axis and the curve can be related to the angle $\Phi$ covered by the radius vector of the particle between a minimum and the subsequent maximum of the radius.

is assigned a curve in phase space defined by (1), containing the complete information on the orbit. Clearly, only values of $u \geq 0$ are physically meaningful, $u = 0$ denotes infinite distance of the particle from the center of force. In the absence of a potential, (1) is the equation of a half circle with radius $\sqrt{2E'}$. Bound orbits correspond to phase space curves completely contained in the $u > 0$ region (cf. Figure 1). While the explicit shape of the phase space portrait is determined by the potential function $V(r)$, some restrictions follow immediately from the structure of the differential equation. As a change of sign of $u'$ and $\theta$ respectively does not effect the equation, the phase space portrait needs to be symmetric about the $u$-axis. Consequently, the orbit exhibits axes of symmetry at angles of $\theta$ where $u'(\theta) = 0$. Assuming the potential to be a continuous, monotonic function of the distance $r$, the number of intersections with the $u$-axis is for bound states equal to two. This corresponds to motion between precisely one maximum and one minimum value of the distance often referred to as the apoapsis and the periapsis of the orbit. In conclusion, orbits are completely determined knowing only the part between a periapsis and the subsequent apoapsis (or vice versa), since the rest can be generated by reflecting about the apsidal vectors (cf. Figure 2).[3],[4]

Taking into account the before stated reflection symmetry, an orbit retracing itself after a finite integer $n$ of revolutions has to align with a periapsis (or an apoapsis with an apoapsis) after $n$

revolutions. For this to happen, the angle $\Phi$ included between periapsis and apoapsis needs to be commensurable with $2\pi$ \(^3\), i.e.

$$m \Phi = n 2\pi \quad m, n \in \mathbb{N} \quad (2)$$

Making use of (1) and the phase space approach presented before, $\Phi$ can be expressed as the partial derivative of the area $A$ enclosed by the phase space curve and the $u$-axis (cf. Figure 1) with respect to $E'$.

$$\Phi = \int_{r_{\min}}^{r_{\max}} \frac{d \theta}{dE'}(r) = \int_{u_{\min}}^{u_{\max}} u' du = \frac{\partial A}{\partial E'} \quad (3)$$

To sum up, whether an orbit is closed can be decided knowing the phase space portrait of the orbit by the necessary and sufficient condition given by (2) combined with (3):

$$\Phi = \frac{\partial A}{\partial E'} \in \pi \mathbb{Q} \quad (4)$$

In general, the value of $\Phi$ for a given potential depends on both the energy and the angular momentum of the particle revolving around the center of force. The object of this paragraph is to show that in the case of a potential imposing closed orbits for all bound states, $\Phi$ needs to be independent of these quantities. Demanding such a case, clearly, $\Phi(E, \ell)$ has to be a rational number

\(^3\) The angle covered by the radius vector of the particle in $n$ revolutions has to contain $\Phi$ a whole number of times.
times \pi for all values of \( E \) and \( \ell \), for which the particle is bound. Thus, varying \( E \) and \( \ell \) continuously, \( \Phi \) could only vary discontinuously in order to fulfill (2). Changing energy or angular momentum infinitesimally would have to cause a finite change of \( \Phi \). Therefore, if one excludes the possibility of jump discontinuities, \( \Phi(E, \ell) \) has to be a constant function, i.e. the same rational number times \( \pi \) for all bound states. [3]

A key step proving Bertrand’s theorem is the analysis of slightly perturbed stable circular orbits. Demanding a potential producing closed orbits for all bound states, clearly also these orbits need to be closed. For a circular orbit \( u' = 0 \) and hence the phase space portrait is a point \((u_0, 0)\) on the \( u \)-axis. As can be seen from (1), all attractive potentials permit circular orbits for certain initial conditions. Applying the above introduced phase space apparatus to orbits deviating slightly from circularity (cf. Appendix A) reveals an elliptic phase space curve with its center at \((u_0, 0)\) corresponding to harmonic oscillations of the particle about the circular orbit. Demanding closedness results in a surprisingly strong restriction for the potential: \( V(r) \) needs to vary as a power of the distance, i.e.

\[
V(r) = kr^s \quad s > -2
\]

Moreover, \( \Phi \) needs to be of the form

\[
\Phi = \frac{\pi}{\sqrt{2 + s}}
\]

(6)

Having reduced the possible functional forms of the potential such drastically, only power laws need to be considered in the following discussion.

Returning to general bound orbits, two cases remain to be analyzed:

i) \( s > 0 \), \( k > 0 \)

ii) \( s < 0 \), \( k < 0 \)

Only in these cases the potential is attractive.\(^4\) Knowing that \( \Phi \) is the same for all initial conditions significantly simplifies the following final step, because it allows to evaluate the integral given by (3) in the limit of very particular values of \( E \) and \( \ell \) for which it is analytically solvable. In the first case, the potential grows to infinity as the distance is increased. Treating orbits in the limit of \( E' \to \infty \) therefore still represents bound states. The phase space portrait in this case turns out to be a half circle with radius \( \sqrt{2E'} \) like in the absence of a potential. \( A \) is thus the area of a quarter of a circle, yielding

\[
A = \frac{\pi}{2} E' \quad \Rightarrow \quad \Phi = \frac{\pi}{2}
\]

(7)

\(^4\) The case where \( s = 0 \) corresponds to a constant potential and therefore doesn’t need to be considered.

\[
\lim_{E' \to 0} \Phi = \frac{\pi}{\frac{2}{2 + s}}
\]

(8)

Comparing with (6), \( s = -1 \) corresponding to a potential \( V = -\frac{|k|}{r} \) and an inverse-square force law. Therefore, the only two potentials for which all bound orbits are closed are the two stated above. [4]

\(^5\) Here, orbits are bound for \( E' < 0 \).
APPENDIX A: CLOSEDNESS OF SLIGHTLY PERTURBED STABLE CIRCULAR ORBITS [4]

Demanding a potential producing closed orbits for all bound states, clearly also slightly perturbed circular orbits need to be closed. For a circular orbit \( u' = 0 \) and hence the phase space portrait is a point \((u_0, 0)\) on the \( u \)-axis. As can be seen from (1), circular orbits are possible for all attractive potentials, if the energy is given by

\[
E'_0 = \frac{1}{2} u_0^2 + \alpha V \left( \frac{1}{u_0} \right). \tag{A1}
\]

A circular orbit is called stable if the effective potential \( W(u) = \frac{1}{2} u^2 + \alpha V(1/u) \) exhibits a minimum at \( u_0 \), i.e.

\[
W'(u_0) = u_0 + \alpha V'(1/u_0) = 0 \quad \text{and} \quad W''(u_0) = 1 + \alpha V''(1/u_0) > 0 \tag{A2}
\]

Slightly increasing the energy for a circular orbit by a small amount \( \delta E' \) and substituting \( u = u_0 + \delta u \), the phase space equation (1) becomes

\[
\frac{1}{2} \left( \frac{d\delta u}{dt} \right)^2 + W(u_0 + \delta u) = E'_0 + \delta E' \tag{A3}
\]

Expanding the potential up to second order in \( u \) around \( u_0 \) and making use of the conditions for stable circular orbits yields the equation of an ellipse with its center at \((u_0, 0)\) enclosing an area given by

\[
A = \frac{\pi \delta E'}{\sqrt{1 + \alpha V''(1/u_0)}}. \tag{A4}
\]

Considering (4), and demanding \( \Phi \) to be a constant rational number times \( \pi \) one has

\[
\Phi = \frac{\pi}{\sqrt{1 + \alpha V''(1/u_0)}} = \text{const.} \tag{A5}
\]

This can only by fulfilled if

\[
1 + \alpha V''(1/u_0) = \text{const.} := 2 + s \tag{A6}
\]

setting the constant equal to \( 2 + s \) for a more convenient formulation in the following equations. (A6) constitutes a second order linear differential equation for the potential with the solution

\[
V(1/u) = ku^{-s} \quad \text{resp.} \quad V(r) = kr^s \quad s > -2. \tag{A7}
\]

Re-substituting into (A5) yields

\[
\Phi = \frac{\pi}{\sqrt{2 + s}}. \tag{A8}
\]

APPENDIX B: NUMERICALLY SOLVING FOR THE ORBIT

To express central force motion in a system of first order ordinary differential equations as required for the fourth order Runge-Kutta method (RK4) and for many other numerical methods, the Hamiltonian formulation is most convenient. The system Hamiltonian, which is identical in value with the total energy of the system, is given by equation B1.

\[
H = \frac{p_r^2}{2m} + \frac{p_\theta^2}{2mr^2} + V(r) \quad \tag{B1}
\]

The required system of first order differential equations is then specified by Hamilton’s equations shown in equation B2 in a compact vector form.

\[
\begin{pmatrix}
\dot{r} \\
\dot{\theta} \\
\dot{p_r} \\
\dot{p_\theta}
\end{pmatrix} =
\begin{pmatrix}
p_r/m \\
p_\theta/mr^2 \\
\frac{V'(r)}{mr^2} - \frac{dV(r)}{dr} \\
0
\end{pmatrix} \tag{B2}
\]

Equation B2 represents a system of effectively three coupled non-linear first order ordinary differential equations which can be treated by a standard RK4 method.